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COMPLETE BASES AND WALLMAN REALCOMPACTIFICATIONS

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ABSTRACT. We study a particular class of separating nest generated intersection rings on a Tychonoff space X, that we call complete bases. They are characterized by the equality $\beta(\upsilon(X, \mathfrak{D})) = \omega(X, \mathfrak{D})$ between their associated Wallman spaces. It is proven that for each separating nest generated intersection ring $\mathfrak{D}$ there exists a unique complete base $\hat{\mathfrak{D}}$ such that $\upsilon(X, \mathfrak{D}) = \upsilon(X, \hat{\mathfrak{D}})$. From this result we obtain a necessary and sufficient condition for the existence of a continuous extension to $\upsilon(X, \mathfrak{D})$ of a real-valued function over X. Some applications of these results to certain inverse-closed subalgebras of $C(X)$ are given.

The word space will refer to Tychonoff spaces. In this paper we consider the Wallman compactification $\omega(X, \mathfrak{D})$ and the Wallman realcompactification $\upsilon(X, \mathfrak{D})$ associated with a given base[2] on a space X. For definitions and basic results the reader is referred to **[1]**, **[9]**, **[10]**. We study the bases $\mathfrak{D}$ that coincide with the trace on X of all zero-sets in its associated space $\upsilon(X, \mathfrak{D})$. These bases, that we call complete, have interesting properties. They are characterized by the relation $\beta(\upsilon(X, \mathfrak{D})) = \omega(X, \mathfrak{D})$.[3] For each base $\mathfrak{D}$ on X there exists a unique complete base $\hat{\mathfrak{D}}$ such that $\upsilon(X, \mathfrak{D}) = \upsilon(X, \hat{\mathfrak{D}})$. The base $\hat{\mathfrak{D}}$ is the largest base with the above property and the smallest complete base on X containing $\mathfrak{D}$.

Frink **[4]** has shown that the real-valued functions over a space X which may be continuously extended to $\omega(X, \mathfrak{D})$ are those which are $\mathfrak{D}$-uniformly continuous. In **[3]** D'Aristotle defined countable $\mathfrak{D}$-uniform continuity and he showed that it is a sufficient but not a necessary condition for the existence of a continuous extension to $\upsilon(X, \mathfrak{D})$ of a real-valued function over X. A necessary and sufficient condition has been obtained by Bentley and Naimpally in **[2**, Theorem 6**]**. We give another condition by means of the base $\hat{\mathfrak{D}}$.

In order to provide examples of noncomplete bases, a general result

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[2]By a base on a space X is meant a separating nest generated intersection ring on X (A. K. Steiner and E. F. Steiner **[10]**). R. A. Alò and H. L. Shapiro **[1]** use the term strong delta normal base.

[3]Two extensions T_1 and T_2 of a space X are said to be equivalent if they are homeomorphic via a map that leaves X pointwise fixed. In this case we write $T_1 = T_2$.

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(Theorem 4) is proven. From this result we derive that the σ-algebra of all Lebesgue measurable sets of the real line R is a noncomplete base for the discrete space R.

In the last section we give some applications of the complete bases to certain inverse-closed subalgebras of $C(X)$ (called algebras), as a consequence of an important relationship between algebras and bases stated in [**10**]. To each algebra A on X a certain natural base $\mathfrak{Z}(A)$ on X is associated. We find that an algebra A on X is $C(Y)$ for some space Y if and only if $\mathfrak{Z}(A)$ is complete. Hence, the examples of noncomplete bases provide examples of algebras that are isomorphic to no $C(Y)$.

Complete bases. When there is no question as to the space X, we will write $\omega(X, \mathcal{D})$ (resp. $\upsilon(X, \mathcal{D})$) as simply $\omega(\mathcal{D})$ (resp. $\upsilon(\mathcal{D})$). The family of all zero-sets in X will be denoted by $Z(X)$. Let Y be a nonempty subset of X. The Q-closure of Y is the set $Q(Y, X)$ of all points $x \in X$ for which every zero-set in $Z(X)$ containing x has a nonempty intersection with Y. The subset Y is Q-dense in X if $Q(Y, X) = X$.

The following result about extension of maps is needed.

THEOREM 1. *Let X be a dense subspace of a space T and let $\mathcal{D}$ be a base on a space Y. A continuous map φ: $X \to Y$ has a continuous extension from T to $\upsilon(\mathcal{D})$ if and only if for any sequence $\{D_n\}_{n=1}^{\infty}$ of sets in $\mathcal{D}$ such that $\bigcap_{n=1}^{\infty} D_n = \varnothing$, we have $\bigcap_{n=1}^{\infty} \mathrm{cl}_T \varphi^{-1}(D_n) = \varnothing$.*

Slight modifications in the proof of Theorem 9.9 in [**11**] show the result.

A base $\mathcal{D}$ on a space X is said to be complete if it coincides with the family $\hat{\mathcal{D}} = \{Z \cap X\colon Z \in Z(\upsilon(\mathcal{D}))\}$. Since $\mathcal{D}$ is the trace on X of all zero-sets in the Wallman compactification $\omega(X, \mathcal{D})$ [**10**, Theorem 2.2], we have $\mathcal{D} \subset \hat{\mathcal{D}}$. An example of a complete base is $Z(X)$. Later, various examples of noncomplete bases will be given.

The following theorem is the main result.

THEOREM 2. *If $\mathcal{D}$ is a base on a space X, then $\upsilon(\hat{\mathcal{D}}) = \upsilon(\mathcal{D})$.*

PROOF. For convenience we write $E = \upsilon(\hat{\mathcal{D}})$ and $F = \upsilon(\mathcal{D})$. Since $\mathcal{D} \subset \hat{\mathcal{D}}$, from Theorem 1 there exists a continuous map ψ from E into F whose restriction to X is the identity. It suffices to prove that ψ is a bijection from E onto F whose inverse is continuous.

Let p be an arbitrary point in F. Then $\{p\} = \bigcap\{Z \in Z(F)\colon p \in Z\}$. Since X is Q-dense in F [**1**, Theorem 5.16], the family $\{Z \cap X\colon p \in Z, Z \in Z(F)\}$, is a $\hat{\mathcal{D}}$-ultrafilter with the countable intersection property. If $q \in \bigcap\{\mathrm{cl}_E(Z \cap X)\colon p \in Z, Z \in Z(F)\}$, then $\psi(q) = p$ and therefore ψ is onto. Let us suppose now that q_1 and q_2 are distinct points of E. There exist $Z_1, Z_2 \in \hat{\mathcal{D}}$ such that $q_i \in \mathrm{cl}_E Z_i$, $i = 1, 2$, and $Z_1 \cap Z_2 = \varnothing$. If $Z_i' \in Z(F)$ and $Z_i' \cap X = Z_i$, then $\psi(q_i) \in \mathrm{cl}_F(Z_i' \cap X) \subset Z_i'$, $i = 1, 2$. As X is Q-dense in F and $Z_1 \cap Z_2 = \varnothing$, we have that $Z_1' \cap Z_2' = \varnothing$ and $\psi(q_1) \neq \psi(q_2)$.

On the other hand, from the Q-density of X in F it follows that for any sequence $\{Z_n\}_{n=1}^{\infty}$ of sets in $\hat{\mathfrak{D}}$ with empty intersection, we have $\bigcap_{n=1}^{\infty} \mathrm{cl}_F Z_n = \varnothing$. From Theorem 1, the identity from $X \subset F$ onto $X \subset E$ has a continuous extension from F into E, which coincides with the inverse of ψ [5, 0.12].

REMARK. In [6] Hager has proved the following interesting result: Let K be a compactification of a space X and let $\mathfrak{D}$ be the family $\{Z \cap X: Z \in Z(K)\}$. Then the Q-closure $Q(X, K)$ is equivalent to the Wallman realcompactification $v(X, \mathfrak{D})$. From Theorem 2 above we obtain $Q(X, K) = v(X, \hat{\mathfrak{D}})$.

The following is an interesting characterization of the complete bases.

COROLLARY 2.1. *A base $\mathfrak{D}$ on a space X is complete if and only if $\beta(v(\mathfrak{D})) = \omega(\mathfrak{D})$.*

PROOF. We write $\mathfrak{S} = \{Z \cap v(\mathfrak{D}): Z \in Z(\omega(\mathfrak{D}))\}$. *Necessity*. Let us prove that $\mathfrak{S} = Z(v(\mathfrak{D}))$. If $Z \in Z(v(\mathfrak{D}))$, then by hypothesis $Z \cap X \in \mathfrak{D}$, and there exists $Z' \in Z(\omega(\mathfrak{D}))$ such that $Z \cap X = Z' \cap X$. Since $v(\mathfrak{D})$ is the Q-closure of X in $\omega(D)$, then $\mathrm{cl}_{v(\mathfrak{D})}(Z' \cap X) = Z' \cap v(\mathfrak{D})$. Therefore $Z' \cap v(\mathfrak{D}) \subset Z \subset \mathrm{cl}_{v(\mathfrak{D})}(Z \cap X) \subset Z' \cap v(\mathfrak{D})$ and $Z = Z' \cap v(\mathfrak{D})$. So $Z \in \mathfrak{S}$, $\mathfrak{S} = Z(v(\mathfrak{D}))$ and consequently $\omega(v(\mathfrak{D}), \mathfrak{S}) = \beta(v(\mathfrak{D}))$. On the other hand, as $\omega(\mathfrak{D}) = \omega(v(\mathfrak{D}), \mathfrak{S})$ [10, Theorem 2.9], it follows that $\beta(v(\mathfrak{D})) = \omega(\mathfrak{D})$.

Sufficiency. By hypothesis $\omega(\mathfrak{D}) = \omega(v(\mathfrak{D}), Z(v(\mathfrak{D})))$ and as $\omega(\mathfrak{D}) = \omega(v(\mathfrak{D}), \mathfrak{S})$, thus $\mathfrak{S} = Z(v(\mathfrak{D}))$ [10, Corollary 2.3] and therefore $\mathfrak{D} = \hat{\mathfrak{D}}$.

COROLLARY 2.2. *The following is true*: (1) *$\hat{\mathfrak{D}}$ is the largest base of X such that $v(\mathfrak{D}) = v(\hat{\mathfrak{D}})$.* (2) *$\hat{\mathfrak{D}}$ is the smallest complete base in X containing $\mathfrak{D}$.*

PROOF. (1) Let $\mathfrak{L}$ be a base in X such that $v(\mathfrak{L}) = v(\mathfrak{D})$. Then $v(\hat{\mathfrak{D}}) = v(\hat{\mathfrak{L}})$. From Corollary 2.1 we have $\omega(\hat{\mathfrak{D}}) = \omega(\hat{\mathfrak{L}})$, therefore $\hat{\mathfrak{D}} = \hat{\mathfrak{L}}$ and $\mathfrak{L} \subset \hat{\mathfrak{D}}$.

(2) Let $\mathfrak{F}$ be a complete base in X containing $\mathfrak{D}$. Then $\mathfrak{D} \subset \hat{\mathfrak{F}}$. From Theorem 1 there exists a continuous map from $v(\hat{\mathfrak{F}})$ into $v(\mathfrak{D})$ whose restriction to X is the identity. By the definition of $\hat{\mathfrak{D}}$ we conclude that $\hat{\mathfrak{D}} \subset \hat{\mathfrak{F}} = \mathfrak{F}$.

Let $\mathfrak{L}$ be a base on a space X. The countable covers of X consisting of sets whose complements are members of $\mathfrak{L}$ form a base for a (compatible) uniform structure on X, denoted by $\mathfrak{U}(\mathfrak{L})$. The countable $\mathfrak{L}$-uniformly continuous functions in the sense of [3] are precisely those (into R) which are uniformly continuous in the uniformity $\mathfrak{U}(\mathfrak{L})$.

THEOREM 3. *Let $\mathfrak{D}$ be a base on a space X. A real-valued function f on X can be continuously extended to $v(\mathfrak{D})$ if and only if it is uniformly continuous in the uniformity $\mathfrak{U}(\hat{\mathfrak{D}})$*

PROOF. *Sufficiency*. It is a consequence of Theorem 2 above and the theorem of [3]. *Necessity*. Given $\varepsilon > 0$, if g is the continuous extension of f to

$v(\mathfrak{D})$, the sets $V_n = \{p \in v(\mathfrak{D}): g(p) \leqslant ((n-1)/3)\varepsilon\} \cup \{p \in v(\mathfrak{D}): g(p) \geqslant ((n+1)/3)\varepsilon, n = 0, \pm 1, \pm 2, \dots\}$ belong to $Z(v(\mathfrak{D}))$. If $O_n = X \sim V_n$, then $\{O_n: n = 0, \pm 1, \pm 2, \dots\}$ is a countable cover of X by $\hat{\mathfrak{D}}$-complements, on each of which the oscillation of f is less than ε, so f is uniformly continuous in the uniformity $\mathfrak{U}(\hat{\mathfrak{D}})$.

THEOREM 4. *Let X be a realcompact space in which every point is a G_δ and let X^* be the set X with a finer completely regular topology. If $\mathfrak{B}$ is a base in X^* containing $Z(X)$, then $\hat{\mathfrak{B}} = Z(X^*)$.*

PROOF. First, let us prove that $X^* = v(\mathfrak{B})$. If $\mathfrak{U}$ is a $\mathfrak{B}$-ultrafilter with the countable intersection property (c.i.p.) the family $\mathfrak{F} = \{Z \in Z(X): Z \in \mathfrak{U}\}$ is a prime $Z(X)$-filter with c.i.p. Let $\mathfrak{V}$ be the (unique) $Z(X)$-ultrafilter with c.i.p. that contains $\mathfrak{F}$ [**11**, Theorem 6.16]. Since X is realcompact there is a point x_0 in X such that $\{x_0\} = \bigcap\{Z: Z \in \mathfrak{V}\}$. By our hypothesis there exists a decreasing sequence $\{Z_n\}_{n=1}^{\infty}$ of zero-sets in $Z(X)$ such that $X \sim \operatorname{int}_X Z_n \in Z(X)$, $n = 1, 2, \dots$, and $\{x_0\} = \bigcap_{n=1}^{\infty} Z_n$. As $\mathfrak{F}$ is prime we have that $Z_n \in \mathfrak{F} \subset \mathfrak{U}$, $n = 1, 2, \dots$, and therefore $x_0 \in \bigcap\{U: U \in \mathfrak{U}\}$. Then $\mathfrak{U}$ is fixed and $X^* = v(\mathfrak{B})$. Since $X^* = v(Z(X^*))$, from Corollary 2.2 we have $\hat{\mathfrak{B}} = Z(X^*)$.

The following example shows that the assumption of $Z(X) \subset \mathfrak{B}$ is essential.

EXAMPLE. Let X be an uncountable discrete space and let $\mathfrak{D}$ be the family $\{M \subset X: M$ is finite or $X \sim M$ is countable$\}$. Thus $\mathfrak{D}$ is a base such that $\omega(\mathfrak{D}) = v(\mathfrak{D})$ is the Alexandroff compactification of X. Since $\omega(\mathfrak{D}) = \beta(v(\mathfrak{D}))$ we have $\mathfrak{D} = \hat{\mathfrak{D}}$, but $\mathfrak{D}$ is not the family $\mathfrak{P}(X)$ of all subsets of X.

COROLLARY 4.1. *Let X be a realcompact space in which every point is a G_δ. Let $\mathfrak{D}$ be a base on X with the discrete topology. If $Z(X) \subset \mathfrak{D}$, then $\hat{\mathfrak{D}} = \mathfrak{P}(X)$.*

Then, the σ-algebra of all Borel sets in R is a noncomplete base of the discrete space R, and also, the σ-algebra of all Lebesgue measurable sets in R.

Subalgebras of $C(X)$. As usual, $C(X)$ will denote the ring of all continuous real-valued functions on a space X. By an algebra on X is meant a subalgebra of $C(X)$ which separates points and closed sets, contains the constants, and is closed under inversion and uniform convergence. If A is an algebra on X and $\mathfrak{Z}(A) = \{Z(f): f \in A\}$, the map $A \to \mathfrak{Z}(A)$ is a one-to-one correspondence between the family of all algebras on X and the family of all bases on X [**10**, Theorem 4.3]. Moreover, if A is an algebra on X isomorphic to $C(Y)$ for some space Y, then $vY = v(\mathfrak{Z}(A))$ and $\beta Y = \omega(\mathfrak{Z}(A))$ [**10**, 4.4]. Therefore:

THEOREM 5. *An algebra A on X is isomorphic to $C(Y)$ for some space Y if and only if $\mathfrak{Z}(A)$ is a complete base.*

It is known that an algebra on X needs not to be $C(X)$, nor any $C(Y)$ **[6]**–**[8]**, **[10]**. The following result shows that such a situation arises in a very large class of standard function algebras used in Topology and Analysis. It is a consequence of Theorems 4 and 5.

THEOREM 6. *Let X be a realcompact space in which every point is a G_δ and let X^* be the set X with a finer completely regular topology. If A is an algebra on X^* containing $C(X)$, then $A = C(X^*)$ or A is not of the form $C(Y)$.*

REMARK. The obvious open problem is to find a constructive method of the completion $\hat{\mathfrak{D}}$. It has to be noted that since the σ-algebra of all Lebesgue measurable sets of the real line R is a noncomplete base of the discrete space R, many usual set operations have to be disregarded.

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