PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 75, Number 1, June 1979

# THE RIGIDITY PROBLEM FOR STABLE SPACES

DONALD W. KAHN

ABSTRACT. This note treats the problem of when the group of homotopy self-equivalences of a space is trivial. For stable spaces, with finitely many nonvanishing homotopy groups, we give a complete solution in an inductive sense. One of the consequences of this result is that for any stable space, with precisely two nonvanishing homotopy groups, the group of self-equivalences is nontrivial.

If $X$ is a (pointed) path connected space, we denote by $G(X)$ the group of homotopy classes of homotopy equivalences from $X$ to itself (all preserving base point). This is the homotopy theorist's analogue of the group of automorphisms of a group, and it has now been studied by many authors. For example, C. Wilkerson **[8]** and D. Sullivan [7] have shown that when $X$ is a simply-connected finite complex, $G(X)$ is a finitely-presented group. Their methods, refining methods in **[4]**, use algebraic groups, and they are not fine enough to get closer to $G(X)$ than a subgroup of finite index, or a quotient by a finite, normal subgroup. On the other hand, there is a space $X$, which at every prime but 3 is a $K(Z, 4)$, but for which $G(X)$ is trivial **[5]**. Since $G(K(Z, 4)) = Z_2$, the integers mod 2, we see that a space $X$ may be equivalent to a space $Y$, at all but one prime, yet $G(X)$ is trivial and $G(Y)$ is not. We shall call a space $X$, for which $G(X)$ is trivial, *rigid*, and we propose to study such spaces–in a stable sense–in this paper.

We shall work in a category whose objects are spaces with finitely-many nonvanishing homotopy groups, in a stable range. Specifically, such a space $X$ shall be endowed with a finite Postnikov tower

$$X = X_k \xrightarrow{\pi_k} \cdots \xrightarrow{\pi_{n+1}} X_n$$

where $\pi_i$ is a principal fibration with fibre $K(\pi_i(X), i)$, $n > 1$, and $k < 2n - 1$. Such a space is infinitely de-loopable, and our continuous maps and homotopies shall be presumed to be infinitely de-loopable also. Base points are assumed but omitted from the notation. We denote this category by $\mathbb{S}_1$. For comparison purposes–at the end of this note–we denote by $\mathbb{S}_2$ the category of simply-connected finite complexes, and stable (in the sense of suspension) homotopy classes of maps.

In a given Postnikov system, if $j > i$, there is a natural homomorphism (see **[2]**)

$$\phi_{j,i}: G(X_j) \to G(X_i),$$

Received by the editors June 29, 1978 and, in revised form, September 5, 1978.
*AMS (MOS) subject classifications* (1970). Primary 55D10; Secondary 55D35, 55D40.
*Key words and phrases*. Homotopy equivalences, stable spaces, Postnikov tower.

which is studied in [3]. The image of $\phi_{m+1,m}$ is precisely known, while the kernel is there related to certain cohomology groups and extensions.

Our idea is now to study the rigidity problem, that is when a space is rigid, by induction. We easily characterize when $K(\pi, n)$, $n > 1$, is rigid (Proposition 2), and then go on to give necessary and sufficient conditions for $\ker(\phi_{m+1,m})$ to be trivial. As a corollary, we may conclude that there is no space $X$ in $\mathcal{S}_1$, with precisely two nonvanishing homotopy groups, which is rigid. In conclusion, we compare this situation with that in the category $\mathcal{S}_2$. We recall (Proposition 3) the well-known result that there is no nontrivial $X$ in $\mathcal{S}_2$, which is rigid. We then show (Proposition 4) that if $X_k$ is a Postnikov term for a finite stable complex $X$, and $X_k$ is rigid, then $\dim(X) \geqslant k$.

Finally, we would like to point out that our results for $\mathcal{S}_1$ do *not* require the homotopy groups of the spaces in question to be finitely-generated, as opposed to many of the other papers in the field. For example, our Proposition 2 actually holds for *all* Abelian groups $\pi$, if we choose $X$ to be cellular.

PROPOSITION 1. *If $X$ is a rigid space, in the category $\mathcal{S}_1$, then the identity map $1_X$ has order* 2 *in the sense of loop multiplication in $X$. In particular, every element in the homotopy or stable homology groups of $X$ has order* 2.

PROOF. Let $[X, X]$ denote the Abelian group of homotopy classes of maps from $X$ to itself, under loop multiplication. Assume that the order of the class of $1_X$ is different from 2. Then $-1_X$ is not homotopic to $1_X$. As $-1_X$ is clearly a homotopy equivalence, $G(X)$ has at least two elements.

For the rest of the proposition, we simply observe that the addition in $[X, X]$ acts additively on homotopy and stable homology (that is homology groups in the stable range).

PROPOSITION 2. *Let $X$ be a space in $\mathcal{S}_1$, with a single, nonvanishing homotopy group, i.e., $X = K(\pi, n)$, $n > 1$. Then $X$ is rigid precisely when $\pi = Z_2$, the integers* mod 2.

PROOF. If $X$ is rigid, then we know that every element of $\pi$ has order 2. I claim that $\pi$ is a $Z_2$-vector space. It is trivial to construct a group homomorphism from a $Z_2$-vector space $V$ onto $\pi$, say $g$: $V \to \pi$. But the kernel of $g$ is trivially checked to be a vector subspace, proving that $\pi$ is the quotient of a vector space by a subspace, and hence, $\pi$ is itself a vector space.

Now, it is well known that $G(K(\pi, n)) = \mathrm{Aut}(\pi)$, the group of automorphisms of $\pi$. But the only $Z_2$-vector space, without a nontrivial automorphism, has dimension 1.

Before stating our main theorem, it will be convenient to specify some notation.

a. $\Sigma X$ means the (reduced) suspension of $X$. $\Sigma f$ means the suspension of the map $f$. $\sigma$: $H^p(Y) \xrightarrow{\approx} H^{p+1}(\Sigma Y)$ is the usual suspension isomorphism in cohomology. $\Omega X$ means loops on $X$.

b. If $Y$ is $(n-1)$-connected, we shall freely write $i_n \in H^n(Y; \pi_n(Y))$ for the fundamental class. If $x \in H^m(Y; \pi)$, write $\phi_x\colon Y \to K(\pi, m)$ for a map such that $\phi_x^*(i_m) = x$.

c. For a fibration in a given Postnikov tower, say

$$X_{n+1} \xrightarrow{\pi_{n+1}} X_n,$$

we shall abbreviate the fibre $K(\pi_{n+1}(X), n+1)$ to $K$. We write the inclusion of the fibre $i\colon K \to X_{n+1}$.

THEOREM. *Let $X$ be a space in $\mathcal{S}_1$, and let*

$$\phi_{m+1,m}\colon G(X_{m+1}) \to G(X_m)$$

*be the homomorphism described above. Then a necessary and sufficient condition that* $\ker(\phi_{m+1,m})$ *be trivial, is that for every cohomology class*

$$x \in H^{m+1}(X_{m+1}; \pi_{m+1}(X)),$$

*for which $i \cdot \phi_x + 1_{X_{m+1}}$ induces an isomorphism on homology in dimension $m+1$, there is a map*

$$\alpha\colon \Sigma X_{m+1} \to X_m$$

*so that $\alpha^*(k^{m+2}) = \sigma(x)$, with $k^{m+2}$ being the k-invariant of the fibration.*

PROOF. We are working in the stable category $S_1$. If we suppose $\{f\} \in \ker(\phi_{m+1,m})$, then the diagram

$$\begin{tikzcd} X_{m+1} \arrow[rr, "f - 1_{X_{m+1}}"] \arrow[dr, "0"'] & & X_{m+1} \arrow[dl, "\pi_{m+1}"] \\ & X_m & \end{tikzcd}$$

is homotopy commutative (0 sends all $X_{m+1}$ to the base point).

By exactness, there is a map $\psi_f\colon X_{m+1} \to K = \pi_{m+1}^{-1}(\text{pt.})$, so that

$$\begin{tikzcd} & K \arrow[dr, "i"] & \\ X_{m+1} \arrow[ur, "\psi_f"] \arrow[rr, "f - 1_{X_{m+1}}"'] & & X_{m+1} \end{tikzcd}$$

is homotopy commutative.

Now, the fibration $\pi_{m+1}$ gives rise to an exact sequence of spaces

$$\cdots \to \Omega X_m \xrightarrow{\gamma} K \xrightarrow{i} X_{m+1} \xrightarrow{\pi_{m+1}} X_m \to \cdots;$$

it is well known that if we identify cohomology of $\Sigma\Omega X_m$ and $X_m$ in the stable range, then $\sigma(\gamma^*(i_{m+1})) = k^{m+2}$, the $k$-invariant.

By exactness, the map $f - 1_{X_{m+1}} \simeq i \cdot \psi_f$ will be null-homotopic (and hence $f \simeq 1_{X_{m+1}}$) precisely when there is a map $\alpha'\colon X_{m+1} \to \Omega X_m$, with

$$\begin{tikzcd} X_{m+1} \arrow[rr, "\alpha'"] \arrow[dr, "\psi_f"'] & & \Omega X_m \arrow[dl, "\gamma"] \\ & K & \end{tikzcd}$$

being homotopy commutative.

If we suspend this diagram and write $\alpha = \Sigma\alpha'$, then stabily we get a homotopy commutative diagram

$$\begin{tikzcd} \Sigma X_{m+1} \arrow[rr, "\alpha"] \arrow[dr, "\Sigma\psi_f"'] & & X_m \arrow[dl, "\Sigma\gamma"] \\ & K(\pi_{m+1}(X), m+2) & \end{tikzcd}$$

On cohomology, $\alpha^*(k^{m+2}) = (\Sigma\psi_f)^*(i_{m+2}) = \sigma(\psi_f^*(i_{m+1}))$.

To complete the proof, we must see that the classes $x$, with the condition given in the statement of the theorem, are precisely those classes of the form $\psi_f^*(i_{m+1})$. But given such a class $x$, $i \cdot \phi_x + 1_{X_{m+1}}$ clearly induces isomorphisms on homology in dimensions through $m + 2$ (recall $H_{m+2}(K) = 0$) and is thus a homotopy equivalence $f$. Then $f - 1_{X_{m+1}} \simeq i \cdot \phi_x$, so we take $\phi_x = \psi_f$ and $\psi_f^*(i_{m+1}) = \phi_x^*(i_{m+1}) = x$. On the other hand, given $\psi_f$–for a homotopy equivalence $f$, with $\{f\} \in \ker(\phi_{m+1,m})$–the class $\psi_f^*(i_{m+1})$ clearly meets the condition of the theorem.

We remark that the theorem is clearly valid for $k(\phi_{j,i})$, $j > i$, when the homotopy groups in dimensions between $i$ and $j$ vanish.

COROLLARY. *If $X$ is a space in $\mathbb{S}_1$ with precisely two nonvanishing homotopy groups, $X$ is not rigid.*

PROOF. By our propositions, we assume the two nonvanishing groups $\pi_i(x)$ and $\pi_j(x)$, $i < j < 2i - 1$, are both direct sums of copies of $Z_2$. We shall prove $\ker(\phi_{j,i})$ is nontrivial.

By [6], the $Z_2$-cohomology of $K(Z_2, i)$ is nonzero in every dimension not less than $i$. In particular, the subgroup

$$H^j\big(K(\pi_i(X), i); \pi_j(X)\big) \subseteq H^j\big(X_j; \pi_j(X)\big)$$

is nontrivial. Let $x \neq 0$ be a class in this subgroup. Consider the map

$$X_j = X \xrightarrow{\phi_x \times 1_X} K\big(\pi_j(X), j\big) \times X \xrightarrow{\bar{\mu}} X,$$

where $\bar{\mu}$ is the action of the fibre. It clearly induces isomorphisms on homotopy groups and is thus a homotopy equivalence. Stably, it is just $i \cdot \phi_x + 1_X$.

We must ask whether there is $\alpha: \Sigma X \to X_i$ with $\alpha^*(k^{j+1}) = \sigma(x) \neq 0$. But

$\Sigma X$ is $i$-connected and $X_i = K(\pi_i(X), i)$, so every $\alpha$ is null-homotopic.

REMARKS. 1. Combining [3] with this Corollary, it is not hard to show that if $X$ is in $\mathcal{S}_1$ and every $k$-invariant is obtained from the fundamental class by a *primary* cohomology operation, then $X$ is *not* rigid.

2. It would be interesting to get information on $G(X)$ for spaces $X$ with 3 or more homotopy groups. If $X$ is rigid, then there is some $j$ with $G(X_j)$ nontrivial, yet $G(X_{j+1})$ is trivial.

We now compare these results with the category of finite complexes $\mathcal{S}_2$. The following is well known.

PROPOSITION 3. *If $X$ is a nontrivial, finite, path connected complex, the group of stable self-equivalences is nontrivial.* (*That $X$ is nontrivial means $X$ does not have the homotopy-type of a point.*)

PROOF. D. Anderson and M. Barratt have proved (unpublished) that the additive stable order of $1_X$ cannot be 2. (This may be shown by looking at the Steenrod operation $Sq^1$ in $X \wedge X$. If the stable order of $1_X$ is 2, the same would be true for $1_{X \wedge X}$. Taking $u \in H^i(X : Z_2)$, with $Sq^1 u \neq 0$, of maximal dimension, we get $Sq^2(u \wedge u) \neq 0$, which quickly leads to a contradiction. Or one may study top dimensional cells of $X$, and copy the proof (as in [1]) that the homotopy groups of $S^n \cup_2 e^{n+1}$ have an element of order 4).

Then just as before, $1_X$ and $-1_X$ represent different elements in our group.

PROPOSITION 4. *Let $Y$ be a connected, finite complex $(i-1)$-connected, $i > 1$, and* $\dim Y < 2i - 1$. *Let $Y_j$ be a Postnikov term for $Y$, $i \leqslant j < 2i - 1$. If $Y_j$ is rigid, then $j \leqslant$* $\dim(Y)$.

PROOF. By Proposition 3, it suffices to show that if $j > \dim(Y) + 1$, $G(Y) = G(Y_j)$.

Clearly, any two homotopy equivalences, which represent the same class on $Y_j$, must be homotopic, for $Y$ and $Y_j$ have the same $j$-type.

On the other hand, the fibration $p$: $Y \to Y_j$ has a section over the $(j+1)$-skeleton $S$: $Y_j^{(j+1)} \to Y$. If $f$: $Y_j \to Y_j$ is a (cellular) homotopy equivalence, we may follow $Y \xrightarrow{p} Y_j \xrightarrow{f} Y_j$ by $S$, to get a homotopy equivalence of $Y$ to itself, which maps to the class of $f$. Thus the natural map $G(Y) \to G(Y_j)$ is also onto.

REMARK. On the negative side of things, this paper shows that nonrigidity is almost everywhere in stable homotopy, and the consequences–such as a plurality of possible $k$-invariants for a space–cannot be avoided.

## REFERENCES

1. P. Hilton, *Introduction to homotopy theory*, Cambridge Univ. Press, Cambridge, 1953.

2. D. W. Kahn, *Induced maps for Postnikov systems*, Trans. Amer. Math. Soc. **107** (1963), 432–450.

3. ______, *The group of homotopy equivalences*, Math. Z. **84** (1964), 1–8.

4. ______, *The group of stable self-equivalences*, Topology **11** (1972), 133–140.

5. ______, *Realization problems for the group of homotopy classes of self-equivalences*, Math. Ann. **220** (1976), 37–46.

6. J.-P. Serre, *Cohomologie* Mod 2 *des complexes de Eilenberg-Mac Lane*, Comment. Math. Helv. **27** (1953), 198–232.

7. D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269–332.

8. C. Wilkerson, *Applications of minimal simplicial groups*, Topology **15** (1976), 111–130.

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455