A THEOREM ON EXTENDING REPRESENTATIONS

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Abstract. We obtain sufficient conditions for a unitary representation of a closed subgroup \( M \) of a separable locally compact group \( G \) to have a unique unitary extension to \( G \). The conditions depend on the behavior of the representation on a closed normal subgroup \( N \) of \( G \) contained in \( M \). We then discuss an application of the theorem in particle physics.

Let \( N \) be a closed normal subgroup of a separable locally compact group \( G \), and let \( M \) be a closed subgroup of \( G \) containing \( N \). Let \( H = \{m \in M : gmg^{-1} \in M \text{ for } g \in G \} \). Assume \( L \) is a representation of \( M \) such that \( L|_N \) is type I, and \( L|_N = \int n(a)\alpha d\mu(\alpha), \mu \) a standard measure on \( \hat{N} \); i.e. there is a \( \mu \) conull set in \( \hat{N} \) which is standard. By the von Neumann selection theorem, there is a cross-section \( c \) from \( \hat{N} \) into the concrete irreducible representations of \( N \) such that \( c \) is Borel on a conull set in \( \hat{N} \). Then

\[
\int n(a)\alpha d\mu(\alpha) = \int n(a)c(\alpha) d\mu(\alpha),
\]

and this gives the primary decomposition of \( L|_N \). \( G \) acts on \( \hat{N} \) by \( \alpha \cdot g = \{R \cdot g : R \in \alpha\} \) where \( (R \cdot g)(n) = R(gng^{-1}) \). We do the transitive case first.

Theorem 1. Assume \( \mu \) is supported on \( a_0 \cdot G \). Assume the stabilizer \( K \) of \( a_0 \) is contained in \( H \) and \( L|_H \cdot g \) is unitarily equivalent to \( L|_H \) for each \( g \in G \). Then \( L \) has a unique unitary extension to \( G \). In fact, there is a representation \( W \) of \( K \) such that \( \text{ind}_K^G W \) is the extension. This extension is irreducible precisely when \( W \) is irreducible.

Remark. \( L = \int_{G/M} \text{ind}_x^{\text{reg}_K} W \cdot x d\bar{\mu}(x) \) where \( \bar{\mu} \) is a quasi-invariant measure on \( G/M \).

Proof. Let \( P \) be the projection valued measure based on the Borel subsets of \( a_0 \cdot G \simeq K \setminus G \) giving the direct integral decomposition \( L|_N = \int n(\alpha)c(\alpha) d\mu(\alpha) \). Since \( L|_H \cdot g \simeq L \) for \( g \in G \), \( L|_N \cdot g \simeq L|_N \) for \( g \in G \); and the argument given by Mackey [4, p. 296], shows \( \mu \) is quasi-invariant and \( n(\alpha \cdot g^{-1}) = n(\alpha) \) for \( \mu \)-a.e. \( \alpha \) for each \( g \). Hence, since \( G \) acts transitively, \( n \) is essentially constant. Therefore

\[
L|_N \simeq n \int c(\alpha) d\mu(\alpha) = n \int c(\alpha) d\mu(\alpha) = n \int \omega \cdot \gamma(Kx) d\mu(Kx)
\]

Received by the editors February 27, 1978 and, in revised form, July 6, 1978.

AMS (MOS) subject classifications (1970). Primary 22D10, 22D30; Secondary 22E70.

1Research supported in part by a Louisiana State Summer Faculty Research Award.

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where \( \omega \in \alpha_0 \) and \( \gamma: K \setminus G \to G \) is a Borel cross-section.

Since \( L_m |_{N} L^{-1}_m = L |_{N} \cdot m \simeq L |_{N} \) and \( n \int \omega \cdot \gamma(Kx) \, d\mu(Kx) \) is the central decomposition of \( L |_{N} \), \( L_m P(E)L^{-1}_m = P(E \cdot m^{-1}) \) for Borel subsets \( E \) of \( K \setminus G \) and \( m \in M \). Furthermore, the orbit space \((K \setminus G)^M \) of \( K \setminus G \) under \( M \) is isomorphic to \( G/M \). In fact \( KxM = x^{-1}KxM = xM \), for \( x^{-1}Kx \subset x^{-1}Hx = H \). Hence the orbit space is standard; and if \( \tilde{\mu} \) is the image of \( \mu \) under the map \( Kx \to xM \), \( \mu \) decomposes uniquely \( \tilde{\mu} \)-a.e. into an “integral of measures”, \( \mu = \int_{G/M} \mu_{xM} \, d\tilde{\mu}(xM) \), where each \( \mu_{xM} \) is a Borel measure on \( K \setminus G \) with \( \mu_{xM}(K \setminus G - KxM) = 0 \) (see [2, Theorem 11]). Since \( \mu \) is quasi-invariant under \( M \), \( \mu_{xM} \) is quasi-invariant under \( M \) for \( \tilde{\mu} \)-a.e. \( xM \).

The decomposition \( \mu = \int_{G/M} \mu_{xM} \, d\tilde{\mu}(xM) \) yields a decomposition of \( P \) into \( \int_{G/M} P(xM) \, d\tilde{\mu}(xM) \) where each \( P(xM) \) is a projection valued measure defined on the Borel subsets of \( xM \) satisfying \( P(xM)(E) = 0 \) iff \( \mu_{xM}(E) = 0 \). Define a projection valued measure \( Q \) on the Borel subsets of \( G/M \) by \( Q(F) = P(\bigcup F) \). Then

\[
L^{-1}_m Q(F)L_m = P\left( \bigcup F \cdot m \right) = P\left( \bigcup F \right) = Q(F).
\]

Therefore \( L \) decomposes into a direct integral over \( G/M \); namely, \( L = \int \Theta L(xM) \, d\tilde{\mu}(xM) \). Furthermore,

\[
L(xM)|_N = n \int_{KxM} \omega \cdot \gamma(Kz) \, d\mu_{xM}(Kz)
\]
and

\[
L(xM)(m^{-1})P(xM)(E)L(xM)(m) = P(xM)(E \cdot m),
\]

\( E \) any Borel subset of \( KxM \). Hence the pair \((L(xM), P(xM))\) forms a transitive system of imprimitivity based on the transitive \( M \) space \( KxM \). Let \( \sigma: G/M \to G \) be a Borel cross-section. By Mackey’s imprimitivity theorem [3, Theorem 6.6], \( L(xM) = \text{ind}_{M}^{G/M} \, \sigma(xM) \) \( W(xM) \), where \( W(xM) \) is a unitary representation of \( \sigma(xM)^{-1}K\sigma(xM) \). One can show \( W(xM)|_N \simeq \omega \cdot \sigma(xM) \).

We consider the representation \( L|_H \). The orbit space \((K \setminus G)^H \) of \( K \setminus G \) under \( H \) is isomorphic to \( H \setminus G \) since \( KxH = KxH^{-1}x = Hx \). Hence \( \mu = \int v_{Hx} \, d\tilde{v}(Hx) \) where \( \tilde{v} \) is the image of \( \mu \) under the map \( Kx \to Hx \) and \( v_{Hx} \) is an \( H \) quasi-invariant measure on \( Hx \). This decomposition, as before, gives a decomposition of \( P \) into \( \int \Theta \, P(Hx) \, d\tilde{v}(Hx) \).

\( \tilde{\mu} \) is the image of \( \tilde{v} \) under the map \( Hx \to xM \). Therefore \( \tilde{v} \) decomposes relative to \( \tilde{\mu} \):

\[
\tilde{v} = \int_{G/M} \tilde{v}_{xM} \, d\tilde{\mu}(xM), \quad \tilde{v}_{xM} \text{ a measure on } H \setminus G \text{ with } \tilde{v}_{xM}(H \setminus G - xM) = 0.
\]

Hence

\[
\mu = \int_{H \setminus G} v_{Hx} \, d\tilde{v}(Hx) = \int_{G/M} \int_{H \setminus G} v_{Hy} \, d\tilde{v}_{xM}(Hy) \, d\tilde{\mu}(xM)
\]

\[
= \int_{G/M} \mu_{xM} \, d\tilde{\mu}(xM).
\]
By uniqueness of decompositions, \( \nu_{xM} = \int_{H \setminus G} \nu_{Hy} \, d\nu_{xM}(Hy) \) for \( \tilde{\nu} \)-a.e. \( xM \). Furthermore, one has \( P(xM) = \int_{H \setminus G} P(Hy) \, d\nu_{xM}(Hy) \).

Define a projection valued measure \( Q^* \) on the Borel subsets of \( H \setminus G \) by \( Q^*(F) = P(\cup F) \). Then \( L_h Q^*(F) L_h^{-1} = Q^*(F) \) for \( h \in H \). Therefore \( L|_H \) decomposes into a direct integral:

\[
L|_H = \int_{H \setminus G} L(Hy) \, d\nu(Hy),
\]

where \( L(Hy)|_N = n_{H|_N} \omega \cdot \gamma(Kz) \, d\nu_{Hy}(Kz) \). Furthermore, since \( P(xM) = \int_{xM} P(Hy) \, d\nu_{xM}(Hy) \), one has

\[
L(xM)|_H = \int_{\sigma(xM)^{-1} K \sigma(xM)} \text{ind}_{\sigma(xM)} W(xM) \, d\nu_{xM}(Hy).
\]

This decomposition of \( (L(xM), P(xM)) \) is precisely the one given by Mackey's subgroup theorem. Note \( KxM \) as an \( M \) space is isomorphic to \( \sigma(xM)^{-1} K \sigma(xM) \setminus M \) and the orbit space of \( KxM \) under \( H \) is isomorphic to the double coset space of \( \sigma(xM)^{-1} K \sigma(xM) \setminus H \) double cosets in \( M \). The correspondence with \( KxM \) carries the double coset \( \sigma(xM)^{-1} K \sigma(xM) \) double cosets in \( M \). Hence the double cosets of \( KxM \) are the \( H \) cosets in \( KxM \). Now let \( \rho: H \setminus G \rightarrow G \) be a Borel cross-section. Then \( \rho|_{KxM} \rightarrow G \) is a cross-section of the \( K \setminus H \) double cosets in \( KxM \) and \( Hy = K \sigma(xM) \sigma(xM)^{-1} \rho(Hy) H \). Therefore

\[
L(Hy) \simeq \int_{\rho(Hy)^{-1} K \sigma(Hy)} W(xM) \cdot \sigma(xM)^{-1} \rho(Hy)
\]

\( \tilde{\nu}_{xM} \)-a.e. \( Hy \). \( \tilde{\nu} \)-a.e. \( xM \). For more details, see the proof of the subgroup theorem [4, p. 227].

We now have

\[
L|_H \simeq \int_{H \setminus G} \text{ind}_{\rho(Hy)^{-1} K \sigma(Hy)} W(yM) \cdot \sigma(yM)^{-1} \rho(Hy) \, d\nu(Hy),
\]

and \( P = \int_{H \setminus G} P(Hy) \, d\nu(Hy) \) where \( P(Hy) \) is the canonical projection valued measure associated with the induced representation \( L(Hy) \). Therefore

\[
L|_H \simeq L|_H \cdot g \simeq \int_{H \setminus G} \text{ind}_{g^{-1} \rho(Hy)^{-1} K \sigma(Hy) g} W(yM) \cdot \sigma(yM)^{-1} \rho(Hy) g \, d\nu(Hy).
\]

Since \( \rho(Hy) g = \rho(Hyg) h \) for some \( h \),

\[
L|_H \simeq \int_{H \setminus G} \text{ind}_{\rho(Hyg)^{-1} K \sigma(Hyg)} W(yM) \cdot \sigma(yM)^{-1} \rho(Hyg) \, d\nu(Hy).
\]

In this last direct integral decomposition, \( P \) decomposes into \( P = \int_{H \setminus G} P(Hy) \cdot g \, d\nu(Hy) \) where \( P(Hy) \cdot g(E) = P(Hy)(E \cdot g^{-1}) \). Changing \( y \rightarrow yg^{-1} \) in the last decomposition yields

\[
L|_H \simeq \int_{H \setminus G} \text{ind}_{\rho(Hy)^{-1} K \sigma(Hy)} W((yg^{-1})M) \cdot \sigma((yg^{-1})M)^{-1} \rho(Hy) \, d\nu(Hy).
\]
and

\[ P = \int \oplus P(H_yg^{-1}) \cdot g \, d\nu(H_y). \]

But

\[ P(H_yg^{-1}) \cdot g = P(H_y). \]

Hence we have two decompositions of \((L|_H, P)\) relative to \(Q^*\). Therefore

\[ \left( \text{ind } W(yg^{-1}M) \cdot \sigma(yg^{-1}M)^{-1} \rho(H_y), P(H_y) \right) \approx \left( \text{ind } W(yM) \cdot \sigma(yM)^{-1} \rho(H_y), P(H_y) \right) \]

for \(\nu\)-a.e. \(H_y\). It follows that

\[ W(yg^{-1}M) \cdot \sigma(yg^{-1}M) \rho(H_y) \approx W(yM) \cdot \sigma(yM)^{-1} \rho(H_y) \]

for \(\nu\)-a.e. \(H_y\) for each \(g \in G\). Hence there is a representation \(W\) of \(K\) such that \(W(xM)\) is unitarily equivalent to the representation \(\sigma(xM)^{-1}k_\alpha(xM) \rightarrow W(k)\) for \(\nu\)-a.e. \(xM\). We then have

\[ L \cong \int_{G/M} \text{ind} \, W \cdot \sigma(xM) \, d\mu(xM). \]

Let \(U = \text{ind}^G_W\). By Mackey's subgroup theorem, \(U\) extends \(L\). \(U\) is unique for if \(U\) is an extension of \(L\), \((U, P)\) is a transitive system of imprimitivity based on \(K \setminus G\). Hence \(U = \text{ind}^G_W\). But then

\[ \left( \text{ind} \, W \cdot \sigma(xM), P(xM) \right) \cong \left( \text{ind} \, W' \cdot \sigma(xM), P(xM) \right) \]

for \(\mu\)-a.e. \(xM\). Hence \(W \cong W'\).

The commuting rings of \(U\) and \(W\) are isomorphic. Hence \(U\) is irreducible when \(W\) is. Q.E.D.

Remark. The stabilizer \(K\) of \(\alpha_0\) is contained in \(H\) if the stabilizer of \(\mu\)-a.e. \(\alpha\) is contained in \(M\).

We next assume \(\mu\) is based on a standard \(G\)-invariant subset \(S\) of \(\hat{N}\).

Theorem 2. Assume \(\mu\) is based on \(S\) and \(G\) acts regularly on \(S\) and \(L|_H \cdot g \cong L|_H\) for \(g \in G\). Then \(L\) has a unique extension from \(M\) to \(G\) provided that the stabilizer of \(\alpha\) is contained in \(M\) for \(\mu\)-a.e. \(\alpha\).

Proof. Let \(S^G\) be the orbit space for \(S\) under \(G\). Let \(\tilde{\mu}\) be the quotient measure. The action is regular if there exists a \(\tilde{\mu}\) conull set \(S_0\) in \(S^G\) such that \(\tilde{S}_0\) is standard. Replacing \(S\) by the inverse image \(S_0\) of \(\tilde{S}_0\) in \(S\), we may
assume $S^G$ is standard. Let $P$ be the projection valued measure defined in the proof of Theorem 1. The measure $\mu$ decomposes relative to $\tilde{\mu}$; $\mu = \int \mu_\xi \, d\tilde{\mu}(\xi)$ where each $\mu_\xi$ is a $G$ quasi-invariant measure based on $\xi$. This leads to a decomposition of both $P$ and $L$: $P = \int P_\xi \, d\tilde{\mu}(\xi)$ and $L = \int L_\xi \, d\tilde{\mu}(\xi)$.

Since the stabilizer of $\alpha$ is contained in $M$ for $\mu$-a.a. $\alpha \in S$, the stabilizer of $\alpha \in \tilde{\xi}$ is contained in $M$ for $\mu_{\tilde{\xi}}$-a.a. $\alpha$ for $\mu_{\tilde{\xi}}$-a.a. $\tilde{\xi}$. Since $L|_H \cdot g \simeq L|_H$, $L^\perp|_H \cdot g \simeq L^\perp|_H$ for $\mu_{\tilde{\xi}}$-a.e. $\tilde{\xi}$. By Fubini’s theorem, for $\mu_{\tilde{\xi}}$-a.e. $\tilde{\xi}$, $L^\perp|_H \cdot g \simeq L^\perp|_H$ a.e. $g$. But $\{ g : L^\perp|_H \cdot g \simeq L^\perp|_H \}$ is a subgroup of $G$. Hence for $\mu_{\tilde{\xi}}$-a.e. $\tilde{\xi}$, $L^\perp|_H$ is invariant under $G$. Hence we may apply Theorem 1 and obtain an extension $R^\perp$ of $L^\perp$ to $G$ for $\mu_{\tilde{\xi}}$-a.e. $\tilde{\xi}$. The extension is a Borel function of $\tilde{\xi}$ and hence $\int R^\perp \, d\tilde{\mu}(\tilde{\xi})$ is an extension of $L$. Since $R^\perp$ is unique a.e. $\tilde{\xi}$, this extension is unique. Q.E.D.

**Remark.** These theorems remain valid for $\sigma$-representations. One can either work in the group extension or modify the above proofs.

As an application of the theorem we show how it is possible to determine the dynamics of a free particle system having time translations which do not commute with spatial translations. For instance the spatial transformations of $\mathbb{R}^3$ may be the Euclidean group $E_3 = \{(x, R) : x \in \mathbb{R}^3 \text{ and } R \in SO(3)\}$ with multiplication defined by $(x, R)(y, S) = (x + Ry, RS).$ Time translation by $t$, however, will have the property $t(x, R) = (e^t x, R)t$. This may occur, for instance, if physical space expands exponentially with time. In any case, for this group a free particle of spin $j$, $j$ a half integer, is a projective representation $U$ such that $U_{(x, R)f(y)} = e^{2\pi i x \cdot y} D_j(R)f(\mathbf{R} - y)$ for $f \in L^2(\mathbb{R}^3, C^{2j+1})$ where $D_j$ is the projective irreducible representation of $SO(3)$ with dimension $2j + 1$. We shall, for convenience, assume $j$ is an integer, for in this case $D_j$ is an ordinary representation. Next note one has $M = E_3$ and $N = \mathbb{R}^3$. Then

$$U_{(0, R)} = (2j + 1) \int_{\mathbb{R}^3} e^{2\pi i x \cdot (y)} \, dx$$

and since

$$\exp(2\pi i x \cdot (y)) \cdot t(0, R) = \exp(2\pi i e^{-t} R^{-1} x \cdot (y)),$$

the orbit of $e_1 = (1, 0, 0)$ is almost all of $\mathbb{R}^3$. The stabilizer $K$ of $e_1$ is $\{(y, R) : Re_1 = e_1, y \in \mathbb{R}^3\}$ while $H = M = E_3$. Hence Theorem 1 applies and we need only find $W$. Note $U_{|E_3} = \int_0^R W(r) \, dr$ where $W(r)$ is the representation of $E_3$ on $L^2(S_r, m_r, C^{2j+1})$, $m_r$ the Lebesgue measure on the sphere of radius $r$, defined by

$$W(r)(t, R)f(s) = e^{2\pi i x \cdot s} D_j(R)f(\mathbf{R} - y).$$

One can show $W(r) = \text{ind}^M_K (W \cdot t) = \text{ind}^M_{Kt^{-1}} W \cdot t$ where $W(x, R) = e^{2\pi i x \cdot s} D_j(R)$ for $(x_1, x_2, x_3, R) \in K$. Hence $U = \text{ind}^G_K W$.

For more information on particle systems having these symmetries, see [1].
REFERENCES


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