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EMBEDDING PHENOMENA BASED UPON DECOMPOSITION THEORY: WILD CANTOR SETS SATISFYING STRONG HOMOGENEITY PROPERTIES[1]

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ABSTRACT. We point out the sharpness of earlier results of McMillan by exhibiting a map of the n-sphere S^n, $n > 5$, onto itself having acyclic but non-cell-like polyhedra as its nondegenerate point inverses and for which the image of the set of nondegenerate point inverses is a Cantor set K. Of necessity, K is wildly embedded, and it has the unusual additional property that every self-homeomorphism of K extends to a self-homeomorphism of S^n.

1. Introduction. According to work of D. R. McMillan, if f is a map of S^n to itself such that the image of the set of nondegenerate point inverses is 0-dimensional, then each point inverse is strongly acyclic over the integers (see [**M**] for definitions) and, in particular, has the integral Čech cohomology of a point [**M**, Lemma 5]; moreover, for the case $n = 3$, each point inverse is cellular [**M**, Corollary 3.5]. We show here that for $n \geqslant 5$ this stronger conclusion of cellularity fails in what is known to be the simplest possible case, in which the image of the nondegenerate elements forms a Cantor set.

This image Cantor set K must be wildly embedded (otherwise, K would be defined by cells in S^n, and the inverse image of the defining cells would also be cells, implying that each nondegenerate point inverse is cellular). As an elementary by-product of its construction, K is seen to possess a symmetry previously undiscovered in wild Cantor sets, for it is strongly homogeneously embedded, meaning that each homeomorphism of K onto itself can be extended to a homeomorphism of S^n to itself. Displaying a weaker form of symmetry, the classical examples of Antoine [A] and Blankenship [**Bl**] are homogeneously embedded in the sense that, for any two points p, q in such examples X, there is a homeomorphism H of S^n to itself for which $H(X) = X$ and $H(p) = q$.

Some profound recent developments concerning decompositions of manifolds support what may appear to be the innocuously easy constructions of this paper. The first of these is due to Cannon [C], who showed that for a

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cell-like decomposition G of an n-manifold M ($n \geqslant 5$) such that the image of the nondegenerate elements is contained in a closed k-dimensional set Y ($2k + 1 \leqslant n$), M/G is a manifold (homeomorphic to M) if and only if M/G satisfies the following disjoint disks property: any two maps of the 2-cell I^2 into M/G can be approximated arbitrarily closely by maps having disjoint images. This has been improved by Edwards [**E**], who obtained the same result with no restriction on the image of the nondegenerate elements beyond the requirement that M/G itself be finite dimensional. The second development, concerning the resolution of singularities in nonmanifolds, is due to Bryant and Lacher [**BL**], who showed that if Y is a generalized n-manifold of dimension $n \geqslant 5$ that is known to be an n-manifold except possibly at points of some 0-dimensional closed subset $S(Y)$, then Y is the cell-like image of an n-manifold. This result also has been improved, by Cannon, Bryant and Lacher [**CBL**], who obtained the same conclusion in case the potential nonmanifold set $S(Y)$ is contained in a closed subset of dimension k, where $2k + 2 \leqslant n$. An explicit consequence of the above needed for applications here is the following theorem.

THEOREM A (CANNON, BRYANT AND LACHER). *Suppose Y is a generalized n-manifold, $n \geqslant 5$, such that*

(1) *Y contains a 0-dimensional closed set $S(Y)$ such that $Y - S(Y)$ is an n-manifold, and*

(2) *Y satisfies the disjoint disks property.*

Then Y is an n-manifold.

As in [**BL**] and [**CBL**], a generalized n-manifold is understood to be an ENR (Euclidean neighborhood retract = a retract of an open subset of some Euclidean space) such that, for each $y \in Y$,

$$H_*(Y, Y - \{y\}; Z) \approx H_*(E^n, E^n - \{0\}; Z).$$

2. The basic construction. McMillan [**M**, p. 959] presents an example of an acyclic but non-cell-like map f of S^n ($n \geqslant 4$) to itself such that the image of the nondegenerate elements is an arc. To a great extent the example described below represents a 0-dimensional version of his.

Throughout the remainder of this paper n will represent a fixed integer greater than 4.

Let M^{n-2} be a compact PL homology $(n-2)$-cell (*an* $(n-2)$-manifold-with-boundary having trivial homology groups but nontrivial fundamental group) and let X' be a PL $(n-3)$-spine for M^{n-2}, that is, $X' \subset \operatorname{Int} M^{n-2}$ and M^{n-2} collapses to X'. Let $N^{n-1} = M^{n-2} \times [-1, 1]$, which then has $X = X' \times \{0\}$ as a spine and for which, in particular, $N^{n-1} - X \approx (\partial N^{n-1}) \times [0, 1)$. Let C be the standard "middle thirds" Cantor set in $I = [0, 1]$. Consider the upper semicontinuous decomposition G of $Q = N^{n-1} \times [-2, 2]$ having $\{X \times \{c\} | c \in C\}$ as its collection of nondegenerate elements. Let Q^*

denote the decomposition space Q/G and $\pi: Q \to Q^*$ the decomposition map.

MAIN LEMMA. *The decomposition space $Q^* = Q/G$ is a compact n-manifold-with boundary.*

PROOF. Clearly the image of ∂Q in Q^* is a collared $(n-1)$-manifold. The argument here will establish that the image Y of Int Q is an n-manifold.

The space Y contains a Cantor set K of possible singular points, K corresponding to the image under π of the nondegenerate elements of G, such that $Y - K$ is an n-manifold. Not only does this mean that Y fulfills condition (1) of Theorem A, it also implies that Y is n-dimensional [**HW**, p. 32].

Next we show that Y is locally contractible. This is obvious for points of $Y - K$. Since each point of K has arbitrarily small (closed) neighborhoods homeomorphic to Q^*, it suffices to prove that Q^* is contractible. The construction guarantees that Q^* deformation retracts to $\pi(X \times I)$, and thus the problem reduces further to proving that $\pi(X \times I)$ is contractible. To do that, we name two auxiliary sets of maps. The first is a set of retractions r_c, defined for $c \in C$, of $\pi(X \times I)$ to $\pi(X \times [c, 1])$ sending $\pi(X \times [0, c])$ to the point $\pi(X \times \{c\})$. Before we name the second, we note that for each component (a, b) of $I - C$, $\pi(X \times [a, b])$ is topologically the suspension of the acyclic polyhedron X and, therefore, is contractible (see [**S**, Exercise 8.D.3, p. 461]). Then the second auxiliary set is a family of contractions, where, for each component (a, b) of $I - C$, ψ_t is a contraction of $\pi(X \times [a, b])$, parametrized by $t \in [a, b]$, such that ψ_a is the identity, $\psi_t(\pi(X \times \{b\}))$ is identically $\pi(X \times \{b\})$, and ψ_b is the constant map to $\pi(X \times \{b\})$. Now we define a contradiction h_t $(t \in I)$ of $\pi(X \times I)$ as

$$h_t(\pi(x, s)) = \begin{cases} \pi(x, s) & \text{if } s > t, \\ r_t\pi(x, s) & \text{if } t \in C, \\ \psi_t r_a(x, s) & * \; * \; * \end{cases}$$

where the convention (* * *) governing the final part of this rule is that $s < t$ and t lies in the component (a, b) of $I - C$.

It follows that the finite dimensional, locally contractible separable metric space Y is an ANR [**H**, Theorem V.7.1] and, therefore, is an ENR [**L**, p. 718]. Moreover, because each $\pi^{-1}(y)$ is acyclic, the Vietoris-Begle mapping theorem [**Br**, Theorem V.6.1] and standard duality theory [**S**, Theorem 6.9.10] yield that

$$\begin{aligned} H_{n-k}(Y, Y - \{y\}) &\approx H_{n-k}(\text{Int } Q, \text{Int } Q - \pi^{-1}(y)) \\ &\approx \bar{H}^k(\pi^{-1}(y)) \\ &\approx H^k(\text{point}) \\ &\approx H_{n-k}(E^n, E^n - \{0\}). \end{aligned}$$

As a result, Y is a generalized n-manifold.

Finally, we turn to condition (2) of Theorem A–the disjoint disks property. We first show that, for any dense subset D of K, each map f of I^2 into Y can be approximated by a map of I^2 into $D \cup (Y - K)$. To do this, choose a triangulation T of I^2 with very small mesh. Approximate f by a map g such that $g(T^{(1)}) \subset Y - \pi(X \times [-2, 2])$ (here $T^{(1)}$ denotes the 1-skeleton of T), which is possible, of course, because $\dim \pi(X \times [-2, 2]) \leq n - 2$. Require this approximation g to be so close to f that, for those 2-simplexes σ of T such that $f(\sigma)$ misses K, $g(\sigma)$ also misses K. In case $f(\sigma) \cap K \neq \varnothing$, modify $g|\sigma$ once more in the following manner: $g|\partial\sigma$ is homotopic to a small loop L near the cone point $\pi(X \times \{d\})$ in the space $\pi(N^{n-1} \times \{d\})$, $d \in D$ (which space is topologically the cone on ∂N^{n-1}), by a homotopy moving points along the images of vertical arcs from $Q = N^{n-1} \times [-2, 2]$ and ranging through a small subset of $Y - \pi(X \times [-2, 2]$; the loop L then is contractible in a small subset of $\pi(N^{n-1} \times \{d\})$. Define $g|\sigma$ as such a contraction of $g|\partial\sigma$.

In order to establish the disjoint disks property, we choose disjoint, dense subsets D_1 and D_2 of K. By the preceding paragraph, given maps f_i of I^2 into Y ($i = 1, 2$), we can approximate them by maps g_i such that $g_i(I^2) \subset D_i \cup (Y - K)$ ($i = 1, 2$). This means that $g_1(I^2)$ and $g_2(I^2)$ intersect only at points of the n-manifold $Y - K$. Consequently, we can exploit traditional general position methods to further adjust the maps g_i, changing things only at points of $g_i^{-1}(Y - K)$, to maps h_i ($i = 1, 2$) such that

$$h_1(I^2) \cap h_2(I^2) = \varnothing.$$

As a consequence of Theorem A, Q^* is an n-manifold-with boundary.

3. The map of S^n to itself.

PROPOSITION 1. *There is a non-cell-like map f of S^n to itself* ($n \geq 5$) *such that the image of the nondegenerate point inverses under f is a Cantor set K.*

PROOF. Crucial to this argument is a fact established in the course of the main lemma that Q^* is contractible.

Form a space S from the disjoint union of Q and Q^* by identifying each point $x \in \partial Q$ with $\pi(x) \in Q^*$, form another space T by doubling Q^* along ∂Q^* (T results from the disjoint union of two copies of Q^* by identifying corresponding boundary points), and name a map f of S onto T such that $f|Q$ acts like π in taking Q onto one of the copies of Q^* and that $f|Q^*$ acts as the identity mapping onto the other copy of Q^* in T. Then the set of nondegenerate point inverses of f coincides with that of π, and its image under f is a Cantor set K in T.

Each of S and T is a closed n-manifold. By a simple Mayer-Vietoris calculation, each has the homology of S^n. Moreover, each is simply connected: since Q^* is contractible, $\pi_1(S)$ is generated by the image of $\pi_1(Q)$, which in turn is generated by the image of $\pi_1(\partial Q)$, and which itself is contained in the (trivial) image of $\pi_1(Q^*)$; it is even more obvious that T is simply connected. Hence, S and T are each topologically equivalent to S^n[N].

4. Properties of the Cantor set.

PROPOSITION 2. *There exists a wildly embedded, strongly homogenously embedded Cantor set K in S^n ($n \geqslant 5$). Furthermore, for each $\varepsilon > 0$ there exists $\delta > 0$ such that every homeomorphism h of K to itself moving points less than δ extends to a homeomorphism H of S^n to itself moving points less than ε and fixed outside the ε-neighborhood of K.*

PROOF. The Cantor set K, of course, is the one determined in §3, where $K \subset Q^* \subset T \approx S^n$. As an alternative to the decomposition theory argument sketched in the introduction that K is wild, consider a map $g: \partial I^2 \to N^{n-1} \times \{2\}$ defining a loop in ∂Q that is not contractible in Q. Since Q^* is contractible, πg extends to a map g^* of I^2 in Q^*. If K were tame, g^* could be adjusted, without changing the map on ∂I^2, to a map g' into $Q^* - K$. This leads to the contradiction that $\pi^{-1}g'$ is a contraction of g in Q.

As an aid for studying the embedding of K in Q^*, we reconsider the source Q as $M^{n-2} \times B$, with B representing the 2-cell $[-1, 1] \times [-2, 2]$, and with $C = \{0\} \times C$ the Cantor set in Int B for which $\pi(X' \times C) = K$. The tameness of Cantor sets in the plane implies that each homeomorphism h^* of C onto itself extends to a homeomorphism H^* of B onto itself fixed on ∂B. Then, given any homeomorphism h of K to itself, one induces a homeomorphism $h^* = \pi^{-1}h\pi$ on C, extends h^* to the promised homeomorphism H^* on B, defines a homeomorphism H on $Q^* = \pi(M^{n-2} \times B)$ as $\pi(Id \times H^*)\pi^{-1}$, and finally extends H to other points of $T \approx S^n$ via the identity. Furthermore, because $C \subset B^2$ satisfies the stronger homogeneity property mentioned in the statement of the proposition, the argument just given shows that $K \subset S^n$ satisfies it as well.

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