EMBEDDING PHENOMENA BASED UPON
DECOMPOSITION THEORY: WILD CANTOR SETS
SATISFYING STRONG HOMOGENEITY PROPERTIES\textsuperscript{1}

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Abstract. We point out the sharpness of earlier results of McMillan by
exhibiting a map of the n-sphere $S^n$, $n \geq 5$, onto itself having acyclic but
non-cell-like polyhedra as its nondegenerate point inverses and for which
the image of the set of nondegenerate point inverses is a Cantor set $K$. Of
necessity, $K$ is wildly embedded, and it has the unusual additional property
that every self-homeomorphism of $K$ extends to a self-homeomorphism of
$S^n$.

1. Introduction. According to work of D. R. McMillan, if $f$ is a map of $S^n$
to itself such that the image of the set of nondegenerate point inverses is
0-dimensional, then each point inverse is strongly acyclic over the integers
(see [M] for definitions) and, in particular, has the integral Čech cohomology
of a point [M, Lemma 5]; moreover, for the case $n = 3$, each point inverse is
cellular [M, Corollary 3.5]. We show here that for $n \geq 5$ this stronger
conclusion of cellularity fails in what is known to be the simplest possible
case, in which the image of the nondegenerate elements forms a Cantor set.

This image Cantor set $K$ must be wildly embedded (otherwise, $K$ would be
defined by cells in $S^n$, and the inverse image of the defining cells would also
be cells, implying that each nondegenerate point inverse is cellular). As an
elementary by-product of its construction, $K$ is seen to possess a symmetry
previously undiscovered in wild Cantor sets, for it is strongly homogeneously
embedded, meaning that each homeomorphism of $K$ onto itself can be
extended to a homeomorphism of $S^n$ to itself. Displaying a weaker form of
symmetry, the classical examples of Antoine [A] and Blankenship [Bl] are
homogeneously embedded in the sense that, for any two points $p, q$ in such
examples $X$, there is a homeomorphism $H$ of $S^n$ to itself for which $H(X) =
X$ and $H(p) = q$.

Some profound recent developments concerning decompositions of mani-

folds support what may appear to be the innocuously easy constructions of
this paper. The first of these is due to Cannon [C], who showed that for a

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cell-like decomposition $G$ of an $n$-manifold $M$ ($n > 5$) such that the image of the nondegenerate elements is contained in a closed $k$-dimensional set $Y (2k + 1 < n)$, $M/G$ is a manifold (homeomorphic to $M$) if and only if $M/G$ satisfies the following disjoint disks property: any two maps of the 2-cell $I^2$ into $M/G$ can be approximated arbitrarily closely by maps having disjoint images. This has been improved by Edwards [E], who obtained the same result with no restriction on the image of the nondegenerate elements beyond the requirement that $M/G$ itself be finite dimensional. The second development, concerning the resolution of singularities in nonmanifolds, is due to Bryant and Lacher [BL], who showed that if $Y$ is a generalized $n$-manifold of dimension $n > 5$ that is known to be an $n$-manifold except possibly at points of some 0-dimensional closed subset $S(Y)$, then $Y$ is the cell-like image of an $n$-manifold. This result also has been improved, by Cannon, Bryant and Lacher [CBL], who obtained the same conclusion in case the potential nonmanifold set $S(Y)$ is contained in a closed subset of dimension $k$, where $2k + 2 < n$. An explicit consequence of the above needed for applications here is the following theorem.

**Theorem A (Cannon, Bryant and Lacher).** Suppose $Y$ is a generalized $n$-manifold, $n > 5$, such that

1. $Y$ contains a 0-dimensional closed set $S(Y)$ such that $Y - S(Y)$ is an $n$-manifold, and
2. $Y$ satisfies the disjoint disks property.

Then $Y$ is an $n$-manifold.

As in [BL] and [CBL], a generalized $n$-manifold is understood to be an ENR (Euclidean neighborhood retract = a retract of an open subset of some Euclidean space) such that, for each $y \in Y$,

$$H_*(Y, Y - \{y\}; Z) \cong H_*(E^n, E^n - \{0\}; Z).$$

2. The basic construction. McMillan [M, p. 959] presents an example of an acyclic but non-cell-like map $f$ of $S^n (n > 4)$ to itself such that the image of the nondegenerate elements is an arc. To a great extent the example described below represents a 0-dimensional version of his.

Throughout the remainder of this paper $n$ will represent a fixed integer greater than 4.

Let $M^{n-2}$ be a compact PL homology $(n - 2)$-cell (an $(n - 2)$-manifold-with-boundary having trivial homology groups but nontrivial fundamental group) and let $X'$ be a PL $(n - 3)$-spine for $M^{n-2}$, that is, $X' \subset \text{Int } M^{n-2}$ and $M^{n-2}$ collapses to $X'$. Let $N^{n-1} = M^{n-2} \times [-1, 1]$, which then has $X = X' \times \{0\}$ as a spine and for which, in particular, $N^{n-1} - X \approx (\partial N^{n-1}) \times [0, 1)$. Let $C$ be the standard “middle thirds” Cantor set in $I = [0, 1]$. Consider the upper semicontinuous decomposition $G$ of $Q = N^{n-1} \times [-2, 2]$ having $\{X \times \{c\}| c \in C\}$ as its collection of nondegenerate elements. Let $Q^*$
denote the decomposition space $Q/G$ and $\pi: Q \to Q^*$ the decomposition map.

**Main Lemma.** The decomposition space $Q^* = Q/G$ is a compact $n$-manifold-with-boundary.

**Proof.** Clearly the image of $\partial Q$ in $Q^*$ is a collared $(n - 1)$-manifold. The argument here will establish that the image $Y$ of Int $Q$ is an $n$-manifold.

The space $Y$ contains a Cantor set $K$ of possible singular points, $K$ corresponding to the image under $\pi$ of the nondegenerate elements of $G$, such that $Y - K$ is an $n$-manifold. Not only does this mean that $Y$ fulfills condition (1) of Theorem A, it also implies that $Y$ is $n$-dimensional [HW, p. 32].

Next we show that $Y$ is locally contractible. This is obvious for points of $Y - K$. Since each point of $K$ has arbitrarily small (closed) neighborhoods homeomorphic to $Q^*$, it suffices to prove that $Q^*$ is contractible. The construction guarantees that $Q^*$ deformation retracts to $\pi(X \times I)$, and thus the problem reduces further to proving that $\pi(X \times I)$ is contractible. To do that, we name two auxiliary sets of maps. The first is a set of retractions $r_c$, defined for $c \in C$, of $\pi(X \times I)$ to $\pi(X \times [c, 1])$ sending $\pi(X \times [0, c])$ to the point $\pi(X \times \{c\})$. Before we name the second, we note that for each component $(a, b)$ of $I - C$, $\pi(X \times [a, b])$ is topologically the suspension of the acyclic polyhedron $X$ and, therefore, is contractible (see [S, Exercise 8.D.3, p. 461]). Then the second auxiliary set is a family of contractions, where, for each component $(a, b)$ of $I - C$, $\psi_t$ is a contraction of $\pi(X \times [a, b])$, parametrized by $t \in [a, b]$, such that $\psi_{a}$ is the identity, $\psi_b(\pi(X \times \{b\}))$ is identically $\pi(X \times \{b\})$, and $\psi_b$ is the constant map to $\pi(X \times \{b\})$. Now we define a contradiction $h_t (t \in I)$ of $\pi(X \times I)$ as

$$h_t(\pi(x, s)) = \begin{cases} 
\pi(x, s) & \text{if } s > t, \\
r_t \pi(x, s) & \text{if } t \in C, \\
\psi_{r_t}(x, s) & \text{else}
\end{cases}$$

where the convention (**) governing the final part of this rule is that $s < t$ and $t$ lies in the component $(a, b)$ of $I - C$.

It follows that the finite dimensional, locally contractible separable metric space $Y$ is an ANR [H, Theorem V.7.1] and, therefore, is an ENR [L, p. 718]. Moreover, because each $\pi^{-1}(y)$ is acyclic, the Vietoris-Begle mapping theorem [Br, Theorem V.6.1] and standard duality theory [S, Theorem 6.9.10] yield that

$$H_{n-k}(Y, Y - \{y\}) \approx H_{n-k}(\text{Int } Q, \text{Int } Q - \pi^{-1}(y)) \approx \overline{H}^k(\pi^{-1}(y)) \approx H^k(\text{point}) \approx H_{n-k}(E^n, E^n - \{0\}).$$

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As a result, $Y$ is a generalized $n$-manifold.

Finally, we turn to condition (2) of Theorem A—the disjoint disks property. We first show that, for any dense subset $D$ of $K$, each map $f$ of $I^2$ into $Y$ can be approximated by a map of $I^2$ into $D \cup (Y - K)$. To do this, choose a triangulation $T$ of $I^2$ with very small mesh. Approximate $f$ by a map $g$ such that $g(T^{(1)}) \subset Y - \pi(X \times [-2, 2])$ (here $T^{(1)}$ denotes the 1-skeleton of $T$), which is possible, of course, because $\dim \pi(X \times [-2, 2]) < n - 2$. Require this approximation $g$ to be so close to $f$ that, for those 2-simplexes $\sigma$ of $T$ such that $f(\sigma)$ misses $K$, $g(\sigma)$ also misses $K$. In case $f(\sigma) \cap K \neq \emptyset$, modify $g|\sigma$ once more in the following manner: $g|\sigma$ is homotopic to a small loop $L$ near the cone point $\pi(X \times \{d\})$ in the space $\pi(N^{-1} \times \{d\})$, $d \in D$ (which space is topologically the cone on $\partial N^{-1}$), by a homotopy moving points along the images of vertical arcs from $Q = N^{-1} \times [-2, 2]$ and ranging through a small subset of $Y - m(X \times [-2, 2])$; the loop $L$ then is contractible in a small subset of $\pi(N^{-1} \times \{d\})$. Define $g|\sigma$ as such a contraction of $g|\sigma$.

In order to establish the disjoint disks property, we choose disjoint, dense subsets $D_1$ and $D_2$ of $K$. By the preceding paragraph, given maps $f_i$ of $I^2$ into $Y$ ($i = 1, 2$), we can approximate them by maps $g_i$ such that $g_i(I^2) \subset D_i \cup (Y - K)$ ($i = 1, 2$). This means that $g_1(I^2)$ and $g_2(I^2)$ intersect only at points of the $n$-manifold $Y - K$. Consequently, we can exploit traditional general position methods to further adjust the maps $g_i$, changing things only at points of $g_i^{-1}(Y - K)$, to maps $h_i$ ($i = 1, 2$) such that

$$h_1(I^2) \cap h_2(I^2) = \emptyset.$$

As a consequence of Theorem A, $Q^*$ is an $n$-manifold-with boundary.

3. The map of $S^n$ to itself.

**Proposition 1.** There is a non-cell-like map $f$ of $S^n$ to itself ($n \geq 5$) such that the image of the nondegenerate point inverses under $f$ is a Cantor set $K$.

**Proof.** Crucial to this argument is a fact established in the course of the main lemma that $Q^*$ is contractible.

Form a space $S$ from the disjoint union of $Q$ and $Q^*$ by identifying each point $x \in \partial Q$ with $\pi(x) \in Q^*$, form another space $T$ by doubling $Q^*$ along $\partial Q^*$ ($T$ results from the disjoint union of two copies of $Q^*$ by identifying corresponding boundary points), and name a map $f$ of $S$ onto $T$ such that $f|Q$ acts like $\pi$ in taking $Q$ onto one of the copies of $Q^*$ and that $f|Q^*$ acts as the identity mapping onto the other copy of $Q^*$ in $T$. Then the set of nondegenerate point inverses of $f$ coincides with that of $\pi$, and its image under $f$ is a Cantor set $K$ in $T$.

Each of $S$ and $T$ is a closed $n$-manifold. By a simple Mayer-Vietoris calculation, each has the homology of $S^n$. Moreover, each is simply connected: since $Q^*$ is contractible, $\pi_1(S)$ is generated by the image of $\pi_1(Q)$, which in turn is generated by the image of $\pi_1(\partial Q)$, and which itself is contained in
the (trivial) image of \( \pi_1(Q^*) \); it is even more obvious that \( T \) is simply connected. Hence, \( S \) and \( T \) are each topologically equivalent to \( S^n[N] \).


**Proposition 2.** There exists a wildly embedded, strongly homogeneously embedded Cantor set \( K \) in \( S^n (n \geq 5) \). Furthermore, for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that every homeomorphism \( h \) of \( K \) to itself moving points less than \( \delta \) extends to a homeomorphism \( H \) of \( S^n \) to itself moving points less than \( \varepsilon \) and fixed outside the \( \varepsilon \)-neighborhood of \( K \).

**Proof.** The Cantor set \( K \), of course, is the one determined in \( \S 3 \), where \( K \subset Q^* \subset T \approx S^n \). As an alternative to the decomposition theory argument sketched in the introduction that \( K \) is wild, consider a map \( g: \partial I^2 \to N^{n-1} \times \{2\} \) defining a loop in \( \partial Q \) that is not contractible in \( Q \). Since \( Q^* \) is contractible, \( \pi g \) extends to a map \( g^* \) of \( I^2 \) in \( Q^* \). If \( K \) were tame, \( g^* \) could be adjusted, without changing the map on \( \partial I^2 \), to a map \( g' \) into \( Q^* - K \). This leads to the contradiction that \( \pi^{-1}g' \) is a contraction of \( g \) in \( Q \).

As an aid for studying the embedding of \( K \) in \( Q^* \), we reconsider the source \( Q \) as \( M^{n-2} \times B \), with \( B \) representing the 2-cell \([-1, 1] \times [-2, 2] \), and with \( C = \{0\} \times C \) the Cantor set in \( \text{Int } B \) for which \( \pi(X' \times C) = K \). The tameness of Cantor sets in the plane implies that each homeomorphism \( h^* \) of \( C \) onto itself extends to a homeomorphism \( H^* \) of \( B \) onto itself fixed on \( \partial B \). Then, given any homeomorphism \( h \) of \( K \) to itself, one induces a homeomorphism \( h^* = \pi^{-1}h\pi \) on \( C \), extends \( h^* \) to the promised homeomorphism \( H^* \) on \( B \), defines a homeomorphism \( H \) on \( Q^* = \pi(M^{n-2} \times B) \) as \( \pi(Id \times H^*)\pi^{-1} \), and finally extends \( H \) to other points of \( T \approx S^n \) via the identity. Furthermore, because \( C \subset B^2 \) satisfies the stronger homogeneity property mentioned in the statement of the proposition, the argument just given shows that \( K \subset S^n \) satisfies it as well.

**References**


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