PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 75, Number 1, June 1979

EMBEDDING PHENOMENA BASED UPON DECOMPOSITION THEORY: AN UNUSUAL CELLULAR 2-CELL[1]

ROBERT J. DAVERMAN

ABSTRACT. We construct a 2-cell D cellularly embedded in the n-sphere S^n, $n > 5$, and locally flatly embedded at each of its interior points, but the boundary of which fails to be weakly flat.

1. Introduction. A k-sphere in S^n is said to be *weakly flat* if its complement is homeomorphic to the complement in S^n of the standardly embedded k-sphere. The origin of this concept is found in work of McMillan [9], who showed that an $(n-1)$-sphere S in S^n, $n \geqslant 5$, is weakly flat if and only if it satisfies the following cellularity criterion: For each open set U containing S there exists an open set V containing S such that each loop in $V - S$ is contractible in $U - S$. Duvall [4] introduced the "weak flatness" terminology and proved that a k-sphere S in S^n, $2 \leqslant k \leqslant n-3$, is weakly flat if and only if it satisfies McMillan's cellularity criterion. The two cases $k = 1$ and $k = n-2$ demand a different homotopy condition: a closed subset X of S^n is said to be *globally* 1-alg if for each open set U containing X there exists an open set V containing X such that each loop in $V - X$ that is null homologous in $V - X$ is also null homotopic in $U - X$. This new condition is a reasonable variation to the cellularity criterion, for in case X is a k-sphere ($k \neq 1$, $n-2$) or a cell-like subset of S^n, one can easily show that X is globally 1-alg if and only if X satisfies the cellularity criterion. Completing the characterization of weak flatness for spheres, Daverman [2] showed that a 1-sphere in S^n, $n \geqslant 5$, is weakly flat if and only if it is globally 1-alg, and Hollingsworth and Rushing [7] did the same for $(n-2)$-spheres S in S^n, $n > 5$, under the necessary additional hypothesis that $S^n - S$ have the homotopy type of S^1.

There is one long-standing puzzle concerning weak flatness. Duvall proved that the boundary of each cellular k-cell, $3 \leqslant k \leqslant n-2$, in S^n is weakly flat, and he raised the related question about the boundary of a cellular 2-cell [4]. Previously McMillan's work [9] had given an affirmative answer for the case of a cellular n-cell ($n \geqslant 5$), and almost immediately after weakly flat $(n-2)$-spheres were characterized, Daverman and Rushing [3] gave an affirmative

Received by the editors July 25, 1978 and, in revised form, September 14, 1978.

AMS (MOS) subject classifications (1970). Primary 57A15, 57A60; Secondary 57A40, 57A45, 57A50.

Key words and phrases. Weakly flat sphere, cellular cell, globally 1-alg, cellularity criterion, homology sphere, double suspension, mapping torus.

[1]Research supported in part by NSF Grant MCS76-07274.

answer for the case of a cellular $(n-1)$-cell. This note answers Duvall's question negatively.

The heart of this paper is the Cannon-Edwards **[1]**, **[5]**, **[6]** solution to the double suspension problem, for purposes here best expressed in the following form: If S is a closed $(n-2)$-manifold with the homology groups of S^{n-2}, then the product of E^1 with the cone over S is an n-manifold. There is sufficient freedom in the construction that, if one prefers, one can consider only those homology spheres S that do bound contractible manifolds, thereby eliminating the difficult problem of realizing this product space as a cell-like decomposition of a manifold.

2. The example. First we outline the steps of the construction. After completing the outline, we supply further details for those steps marked with an asterisk. Throughout this section n denotes a fixed integer greater than 4.

Step 1*. Find a PL homology $(n-2)$-sphere S with nontrivial fundamental group π that can be trivialized by adding relations given by some inner automorphism ψ of π (more precisely, the normal closure of $\{g^{-1}\psi(g) | g \in \pi\}$ is π itself).

Step 2*. Determine a PL homeomorphism f of S to itself leaving a neighborhood of some $p \in S$ pointwise fixed and for which the induced homeomorphism on fundamental groups is ψ.

Step 3. Form the cone cS on S, produce a homeomorphism F of cS to itself by coning over f, and construct the mapping torus T of F. Explicitly, T issues from $(cS) \times [0, 1]$ by identifying each point $(x, 0)$ with $(F(x), 1)$. According to the double suspension theorem of Cannon-Edwards **[1]**, **[5]**, **[6]**, T is a compact n-manifold-with-boundary.

Step 4. Observe that the simple closed curve J in ∂T corresponding to $\{p\} \times [0, 1]$ has a tubular neighborhood homeomorphic to $B^{n-1} \times S^1$. Add a 2-handle H to T along this neighborhood, with J as the attaching sphere, to determine a new n-manifold-with-boundary M.

Step 5*. Specify a 2-cell D in Int M as the core of the 2-handle H plus the annulus corresponding to (cone over p) $\times [0, 1]$, after identification, in T. The first claim is that $D \subset M$ satisfies the cellularity criterion, and, therefore, has a neighborhood embeddable in S^n. The second claim is that the image of Bd D under such an embedding cannot be weakly flat.

Now for some details. In case $n = 5$, a perfectly good homology 3-sphere S is the Mazur example **[8]**, the fundamental group π of which has the presentation

$$\left\{x, y \,|\, y^4 = (xy)^2 y (xy)^2, (xy)^2 = y(xy)^2 x^{-1}\left(y^{-1}x^{-1}\right)^2\right\},$$

but any other example with 2-generator group would work equally well. For the inner automorphism ψ on π induced by one of these generators, say y, the normal closure N of the element $x^{-1}\psi(x) = x^{-1}y^{-1}xy$ equals π, since π/N, being abelian, must be trivial. In case $n > 5$, a homology $(n-2)$-sphere S' with fundamental group π, as above, can be given by letting B denote the complement of the interior of a PL 3-cell in S and by defining S' as the double of $B \times I^{n-5}$ (that is, S' equals the closed manifold obtained from two copies of $B \times I^{n-5}$ by identifying corresponding boundary points).

To determine the appropriate homeomorphism f on S, simply transport the base point s of $\pi_1(S, s)$ around the element giving rise to ψ while keeping a large open set pointwise fixed.

Before showing that D satisfies the cellularity criterion, we prove that ∂M is simply connected. Clearly $\pi_1(\partial M)$ results from $\pi_1(\partial T)$ by killing the group element determined by J. The group $\pi_1(\partial T)$ is the semidirect product of $\pi_1(S)$ and the integers Z, represented by adding an extra generator v (corresponding to J) to $\pi_1(S)$, subject to the additional relations $v \cdot \alpha \cdot v^{-1} = f_*(\alpha)$ for each $\alpha \in \pi_1(S)$. Killing v reduces the computation to adding all relations $\alpha = f_*(\alpha)$ for each $\alpha \in \pi_1(S)$ to $\pi_1(S)$, and by construction of ψ and f_*, this results in the trivial group.

The argument above is the key to the geometry. With it we can readily establish the two principal claims.

Claim 1. *D satisfies the cellularity criterion in M.*

It is a direct consequence of the description of M as a simplicial (but not PL!) regular neighborhood of D that D has arbitrarily small neighborhoods M' simplicially isomorphic to M, for which then $M' - D \approx \partial M \times [0, 1)$ is simply connected, implying that D satisfies the cellularity criterion. One might note that doubling M produces a homotopy n-sphere, necessarily S^n [**10**], and quickly embeds M itself in S^n.

Claim 2. *For* Bd $D \subset M \subset S^n$, Bd D *fails to be weakly flat.*

Here $T \subset M \subset S^n$. Then Bd D is contained in T in such a way that the infinite cyclic cover T' of T is the space $(cS) \times E^1$, in which the set X covering Bd D is the line given by the product of the cone point with E^1. Then $T' - X \approx S \times [0, 1) \times E^1$, and X fails to be globally 1-alg in T'. It follows easily that Bd D cannot be globally 1-alg in T, so Bd $D \subset S^n$ cannot be weakly flat [**2**].

References

1. J. W. Cannon, *Shrinking cell-like decompositions of manifolds. Codimension three* (to appear).

2. R. J. Daverman, *On weakly flat 1-spheres*, Proc. Amer. Math. Soc. **38** (1973), 207–210.

3. R. J. Daverman and T. B. Rushing, *Weak flatness criteria for codimension 2 spheres in codimension 1 manifolds*, General Topology and Appl. **6** (1976), 101–115.

4. P. F. Duvall, Jr., *Weakly flat spheres*, Michigan Math. J. **16** (1969), 117–124.

5. R. D. Edwards, *Suspensions of homology spheres* (to appear).

6. ______, *Approximating certain cell-like maps by homeomorphisms* (to appear).

7. J. G. Hollingsworth and T. B. Rushing, *Homotopy characterizations of weakly flat codimension 2 spheres*, Amer. J. Math. **98** (1976), 385–394.

8. B. Mazur, *A note on some contractible 4-manifolds*, Ann. of Math. (2) **73** (1961), 221–228.

9. D. R. McMillan, Jr., *A criterion for cellularity in a manifold*, Ann. of Math. (2) **79** (1964), 327–337.

10. M. H. A. Newman, *The engulfing theorem for topological manifolds*, Ann. of Math. (2) **84** (1966), 555–571.

Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37916