PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 75, Number 2, July 1979

SUBDIRECTLY IRREDUCIBLE ALGEBRAS IN REGULAR, PERMUTABLE VARIETIES[1]

W. TAYLOR

ABSTRACT. If A is a finite algebra in a regular, permutable variety (of finite type), then the variety generated by A either contains an infinite subdirectly irreducible algebra or contains only finitely many subdirectly irreducible algebras. We conjecture that the hypothesis "regular and permutable" cannot be fully removed.

Quackenbush asked [7] whether there exists a finite algebra $\mathfrak{A}$ such that $\mathcal{V} = \mathbf{HSP}\,\mathfrak{A}$ has infinitely many finite subdirectly irreducible (s.i.) algebras but no infinite s.i. algebras. He also asked, "Can it be of finite type; can it be a groupoid, semigroup or group?" Except for groups (settled here, and probably essentially known all along), these questions remain open. Baldwin and Berman [2] relaxed the question to that of the existence of a locally finite $\mathcal{V}$ of finite type with arbitrarily large finite s.i.'s but with no infinite s.i. They were able to find such $\mathcal{V}$ which are locally finite but not of finite type, and also of finite type but not locally finite. And recently Baldwin [1] came close: his $\mathcal{V}$ is locally finite and of finite type, has large finite s.i.'s and exactly *one* infinite s.i.

Here we show that *there is no such $\mathfrak{A}$ in a variety of finite type which is congruence-regular and congruence-permutable*. (See the theorem at the end of this article; it was announced in [9].) We thank K. Baker, J. Baldwin, R. Freese, W. Lampe and E. Nelson for valuable conversations about this material.

We will assume that *congruence-permutability* is well known to the reader (see e.g. [4, pp. 119 and 172]). And $\mathcal{V}$ is *congruence-regular* iff every congruence on every $\mathfrak{A} \in \mathcal{V}$ is determined by any one of its blocks. The main fact we will need about congruence-regularity [3], [5], [10] is that it is equivalent to the existence of $\mathcal{V}$-terms $P_j(x, y, z)$ $(1 \leqslant j \leqslant 2N)$ and $G_i(x, y, z, w)$ $(1 \leqslant i \leqslant N)$ such that $\mathcal{V}$ obeys the following identities:

$$P_j(x, x, z) = z \qquad (1 \leqslant j \leqslant 2N),$$
$$x = G_1(x, y, z, P_1(x, y, z)),$$
$$G_1(x, y, z, P_2(x, y, z)) = G_2(x, y, z, P_3(x, y, z)),$$

Received by the editors October 2, 1978.
AMS (MOS) subject classifications (1970). Primary 08A15.
[1]Research supported in part by N.S.F. Grant MCS 76-06558-A01.

0002-9939/79/0000-0301/$02.25

$$G_2(x, y, z, P_4(x, y, z)) = G_3(x, y, z, P_5(x, y, z)),$$

$$\vdots$$

$$G_N(x, y, z, P_{2N}(x, y, z)) = y.$$

We will use these identities for our first lemma, which refines Mal'tsev's well known description of principal congruence relations [4, p. 54]. Let us suppose that each $\mathcal{V}$-term σ has been assigned an (integer-valued) rank $\#\sigma > 1$ with the usual proviso that $\#F(\sigma_1, \ldots, \sigma_n) >$ each $\#\sigma_i$ (cf. [4, pp. 40–41]). For an algebraic function $f\colon A \to A$, define $\#f$ to be the smallest value of $\#\sigma$ for $f(z) = \sigma(z, a_1, \ldots, a_n)$ (each $a_i \in A$). Define $M_1 = \max\{\#G_i\colon 1 < i < N\}$, and

$M_2 =$ the maximum rank of any term

$$P_i(G(x_1, \ldots, x_{k-1}, \sigma, x_{k+1}, \ldots, x_n), v, w) \qquad (1 < i < 2N)$$

for G any fundamental operation and σ any term of rank $< M_1$. (The numbers M_1 and M_2, of course, depend on the variety $\mathcal{V}$, which for convenience we will keep fixed throughout this discussion.)

Let us now describe the principal congruence $\theta(a, b)$ for $a,b \in \mathfrak{A} \in \mathcal{V}$. Letting f and h denote algebraic functions of one variable, we recursively define

$$A_0 = \{b\},$$
$$A_{i+1} = \{h(c)\colon c \in A_i, h(a) = a, \#h < M_2\}$$

and finally,

$$\theta_0(a, b) = \left\{(f(a), f(c))\colon c \in \bigcup_{i=1}^{\infty} A_i, \#f < M_1\right\}.$$

LEMMA 1. *$\theta(a, b)$ is the transitive-symmetric hull of $\theta_0(a, b)$.*

PROOF. It is clearly enough to see that this hull is closed under the actions of OAF's, i.e. algebraic functions $g\colon A \to A$ formed by freezing all but one of the places in one of the operations G of $\mathfrak{A}$. In fact it is obviously enough to consider arbitrary $(f(a), f(c)) \in \theta_0(a, b)$ and show that $(gf(a), gf(c))$ is in the transitive-symmetric hull of $\theta_0(a, b)$. By definition of $\theta_0(a, b)$, we have $c \in A_i$ for some i, and $\#f < M_1$. Now consider the algebraic functions

$$h_j(z) = P_j(gfz, gfa, a) \qquad (1 < j < 2N).$$

Clearly $\#h_j < M_2$, and $h_j(a) = a$, by our identities; thus each $h_j(c) \in A_{i+1}$. Next consider the algebraic functions

$$f_i(z) = G_i(gf(c), gf(a), a, z) \qquad (1 < i < N).$$

It is clear that $\#f_i < M_1$, and thus we have $(f_i(a), f_i h_j(c)) \in \theta_0(a, b)$; in particular, the following pairs are in $\theta_0(a, b)$:

$$(f_1a, f_1h_1c), \quad (f_1a, f_1h_2c),$$
$$(f_2a, f_2h_3c), \quad (f_2a, f_2h_4c),$$
$$\vdots$$
$$(f_Na, f_Nh_{2N-1}c), \quad (f_Na, f_Nh_{2N}c).$$

In the identities stated above, if we now substitute $gf(c)$ for x, $gf(a)$ for y, and a for z, we obtain

$$\begin{aligned} gfc &= f_1h_1c, \\ f_1h_2c &= f_2h_3c, \\ f_2h_4c &= f_3h_5c, \\ &\vdots \\ f_Nh_{2N}c &= gfa. \end{aligned}$$

Combining these equalities with our list of pairs from $\theta_0(a, b)$, we easily obtain that (gfa, gfc) is in the transitive-symmetric hull of $\theta_0(a, b)$. □

LEMMA 2. *There exists a function ν: $\omega \to \omega$ (depending only on V, which is assumed congruence-regular) with the following property. If $a,b,c,d \in \mathfrak{A} \in \mathcal{V}$ and $(c, d) \in \theta(a, b)$, then (c, d) is in the transitive-symmetric hull of a set of M pairs $(f(a), f(b))$, with each f an algebraic function of rank $\leq \nu(M)$, where M denotes the power of the smallest congruence class containing a, b, c and d.*

PROOF. Consider the representation in Lemma 1. It is clear that $A_M = A_{M+1} = \dots$, since a sequence $b, f_1(b), f_2f_1(b), f_3f_2f_1(b), \dots$ with each $f_i(a) = a$, must repeat itself at least once in M steps, and hence can be shortened if it is longer than M. This obviously places a bound on the ranks of algebraic functions required. And the length of the necessary transitive chain is obviously limited by M. □

Our next lemma is well known for groups (see [6, 51.23, p. 146]), limiting the orders of chief factors in varieties generated by a finite group. We have heard that the general case was previously discovered by J. B. Nation.

LEMMA 3. *If $\mathfrak{A}$ is finite and* **HSP** *$\mathfrak{A}$ is congruence-permutable, then every finite $\mathfrak{B} \in$* **HSP** *$\mathfrak{A}$ has a strictly increasing sequence of congruence blocks $B_0 < B_1 < \cdots < B_k = \mathfrak{B}$, with $|B_i| \leq |\mathfrak{A}|^i$ for each $i \leq k$.*

PROOF. Let us have $\mathfrak{B} = \mathfrak{C}/\theta$ with $\mathfrak{C} \subseteq \mathfrak{A}^m$. We define congruences θ_j $(0 \leq j \leq m)$ on $\mathfrak{C}$ as follows:

$$c\,\theta_j\,c' \leftrightarrow c_i = c_i' \qquad (i = j + 1, j + 2, \dots, m).$$

(And thus $\Delta_{\mathfrak{C}} = \theta_0 \subseteq \theta_1 \subseteq \cdots \subseteq \theta_m = \mathfrak{C}^2$.) Let us look at one $(\theta \vee \theta_j)$-block U, and count the number of $(\theta \vee \theta_{j-1})$-blocks which it contains. Fix $c \in U$. Congruence-permutability tells us that for each $y \in U$ we have $c\,\theta_j\,z\,\theta\,y$ for some $z \in \mathfrak{C}$. Suppose we also have $c\,\theta_j\,z'\,\theta\,y'$ with z and z' agreeing in their jth coordinate; then the definition of θ_j tells us that $z\ \theta_{j-1}\ z'$, and hence $y(\theta \vee \theta_{j-1})y'$. Thus the $(\theta \vee \theta_{j-1})$-block of $y \in U$ is determined by the jth coordinate of z, which can take only $|\mathfrak{A}|$ values. Hence each $(\theta \vee \theta_j)$-block contains at most $|\mathfrak{A}|$ $(\theta \vee \theta_{j-1})$-blocks. Regarding each $\theta \vee \theta_j$ as a congruence ψ_j on $\mathfrak{B} = \mathfrak{C}/\theta$, we have the corresponding property: each ψ_j-block contains at most $|\mathfrak{A}|$ ψ_{j-1}-blocks.

It is now easy to construct the desired sequence of congruence blocks; we start at the top and work downward. Define $U_0 = B$ and U_{i-1} = the largest of all ψ_j-blocks (for any j) which are proper subsets of U_i. Since B is finite and $\psi_0 = \Delta_B$, the process terminates at some singleton U_k. Renumbering upward *via* $B_i = U_{k-i}$ finishes the construction. □

In the spirit of [2] and [8] we define a *congruence formula* to be any positive 4-ary formula $\varphi(\cdot, \cdot, \cdot, \cdot)$ obeying $\forall yz[\exists x\varphi(x, x, y, z) \rightarrow y = z]$. It is shown in [8] that for $a,b \in \mathfrak{A}$, the principal congruence $\theta(a, b)$ consists of all pairs (c, d) for which $\mathfrak{A} \models \varphi(a, b, c, d)$ for some congruence formula φ. (Our proof in [8] was based on Mal'tsev's description mentioned above; a direct proof is even easier: just show that all such (c, d) form a congruence relation.) We now refine this congruence-formula description for regular and permutable varieties generated by a finite algebra.

LEMMA 4. *If $\mathcal{V}$ = **HSP** $\mathfrak{A}$ (with $\mathfrak{A}$ finite) is regular and permutable, then there exists a sequence $\varphi_1, \varphi_2, \ldots$ of congruence formulas with the following property. The elements of every finite s.i. algebra $\mathfrak{B} \in \mathcal{V}$ can be arranged in a sequence $b_0, b_1, b_2, \ldots$ such that $\mathfrak{B} \models \varphi_j(b_i, b_j, b_0, b_1)$ whenever $i < j$.*

PROOF. Take $B_0 \subset B_1 \subset B_2 \subset \ldots$ as provided by Lemma 3. By congruence-regularity, the minimum congruence θ on $\mathfrak{B}$ contains a pair (b_0, b_1) with $b_0, b_1 \in B_1$ and $b_0 \neq b_1$. Now simply choose the sequence so that for each j, $b_j \in B_j$. Certainly for $i < j$ we have $(b_0, b_1) \in \theta(b_i, b_j)$. By Lemma 2, (b_0, b_1) is in the transitive-symmetric hull of at most $|\mathfrak{A}|^j$ pairs $(f(b_i), f(b_j))$, with each f an algebraic function of rank $< \nu(|\mathfrak{A}|^j)$. It should now be clear how we build the formula φ_j. We write down *all* "chains" linking b_0 with b_1 with $< |\mathfrak{A}|^j$ links composed of pairs $(f(b_i), f(b_j))$ (there are a finite number), form their disjunction, and finally existentially quantify over all parameters appearing in the terms designating algebraic functions. □

Notice that the conclusion of this lemma involves a weakened version of Baldwin and Berman's notion [2] of *definable principal congruences*.

THEOREM. *If $\mathfrak{A}$ is a finite algebra with **HSP** $\mathfrak{A}$ congruence-regular and congruence-permutable, and if **HSP** $\mathfrak{A}$ contains arbitrarily large finite subdirectly irreducible algebras, then it contains an infinite subdirectly irreducible algebra.*

PROOF. Let $\mathfrak{B}_i$ be s.i. with $|\mathfrak{B}_i| > i$ $(i > 1)$. Write $\mathfrak{B}_i = \{b_{i0}, b_{i1}, b_{i2}, \ldots\}$ according to Lemma 4. Let $\mathfrak{B}$ be a nonprincipal ultraproduct of $\langle \mathfrak{B}_i : i \in I \rangle$

and take $b_0, b_1, b_2, \ldots \in \mathfrak{B}$ such that for each j, the ith coordinate of b_j is b_{ij} (a.e. in i). Łoś' theorem tells us that $\mathfrak{B} \models \varphi_j(b_i, b_j, b_0, b_1)$ whenever $i < j$ where φ_j is as in Lemma 4. Thus all b_j must be distinct in $\mathfrak{B}/\theta$ whenever $(b_0, b_1) \notin \theta$, and hence such a quotient $\mathfrak{B}/\theta$ is always infinite. If we take θ to be a maximal congruence separating b_0 and b_1, then $\mathfrak{B}/\theta$ is infinite and s.i. □

References

1. J. T. Baldwin, *The number of subdirectly irreducible algebras in a variety*. II, Algebra Universalis (to appear).

2. J. T. Baldwin and J. Berman, *The number of subdirectly irreducible algebras in a variety*, Algebra Universalis **5** (1975), 379–389.

3. B. Csákány, *Characterizations of regular varieties*, Acta Sci. Math. (Szeged) **31** (1970), 187–189.

4. G. Grätzer, *Universal algebra*, Van Nostrand, Princeton, N. J., 1968.

5. ______, *Two Mal'cev-type theorems in universal algebra*, J. Combinatorial Theory **8** (1970), 334–342.

6. H. Neumann, *Varieties of groups*, Springer-Verlag, Berlin, 1967.

7. R. W. Quackenbush, *Equational classes generated by finite algebras*, Algebra Universalis **1** (1971), 265–266.

8. W. Taylor, *Residually small varieties*, Algebra Universalis **2** (1972), 33–53.

9. ______, *Residual finiteness in congruence-regular and permutable varieties*, Notices Amer. Math. Soc. **25** (1978), A-350.

10. R. Wille, *Kongruenzklassengeometrien*, Lecture Notes in Math., vol. 113, Springer-Verlag, Berlin, 1975.

Department of Mathematics, University of Colorado, Boulder, Colorado 80309