SUBDIRECTLY IRREDUCIBLE ALGEBRAS IN REGULAR, PERMUTABLE VARIETIES

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Abstract. If \( A \) is a finite algebra in a regular, permutable variety (of finite type), then the variety generated by \( A \) either contains an infinite subdirectly irreducible algebra or contains only finitely many subdirectly irreducible algebras. We conjecture that the hypothesis "regular and permutable" cannot be fully removed.

Quackenbush asked [7] whether there exists a finite algebra \( \mathfrak{A} \) such that \( \mathcal{V} = \text{HSP} \mathfrak{A} \) has infinitely many finite subdirectly irreducible (s.i.) algebras but no infinite s.i. algebras. He also asked, "Can it be of finite type; can it be a groupoid, semigroup or group?" Except for groups (settled here, and probably essentially known all along), these questions remain open. Baldwin and Berman [2] relaxed the question to that of the existence of a locally finite \( \mathcal{V} \) of finite type with arbitrarily large finite s.i.'s but with no infinite s.i. They were able to find such \( \mathcal{V} \) which are locally finite but not of finite type, and also of finite type but not locally finite. And recently Baldwin [1] came close: his \( \mathcal{V} \) is locally finite and of finite type, has large finite s.i.'s and exactly one infinite s.i.

Here we show that there is no such \( \mathfrak{A} \) in a variety of finite type which is congruence-regular and congruence-permutable. (See the theorem at the end of this article; it was announced in [9].) We thank K. Baker, J. Baldwin, R. Freese, W. Lampe and E. Nelson for valuable conversations about this material.

We will assume that congruence-permutability is well known to the reader (see e.g. [4, pp. 119 and 172]). And \( \mathcal{V} \) is congruence-regular iff every congruence on every \( \mathfrak{A} \in \mathcal{V} \) is determined by any one of its blocks. The main fact we will need about congruence-regularity [3], [5], [10] is that it is equivalent to the existence of \( \mathcal{V} \)-terms \( P_j(x, y, z) \) (\( 1 < j < 2N \)) and \( G_i(x, y, z, w) \) (\( 1 < i < N \)) such that \( \mathcal{V} \) obeys the following identities:

\[
P_j(x, x, z) = z \quad (1 < j < 2N),
\]

\[
x = G_1(x, y, z, P_1(x, y, z)),
\]

\[
G_1(x, y, z, P_2(x, y, z)) = G_2(x, y, z, P_3(x, y, z)).
\]

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We will use these identities for our first lemma, which refines Mal'tsev's well known description of principal congruence relations \([4, \text{p. 54}]\). Let us suppose that each \(\forall\)-term \(\sigma\) has been assigned an (integer-valued) rank \(#\sigma > 1\) with the usual proviso that \(#F(\sigma_1, \ldots, \sigma_n) > \#\sigma_i\) (cf. \([4, \text{pp. 40–41}]\)). For an algebraic function \(f: A \to A\), define \(#f\) to be the smallest value of \(#\sigma\) for \(f(z) = \sigma(z, a_1, \ldots, a_n)\) (each \(a_i \in A\)). Define \(M_1 = \max\{#G_i: 1 < i < N\}\), and

\[
M_2 = \text{the maximum rank of any term}
\]

\[
P_i(G(x_1, \ldots, x_{k-1}, \sigma, x_{k+1}, \ldots, x_n), v, w) \quad (1 < i < 2N)
\]

for \(G\) any fundamental operation and \(\sigma\) any term of rank \(< M_1\). (The numbers \(M_1\) and \(M_2\), of course, depend on the variety \(V\), which for convenience we will keep fixed throughout this discussion.)

Let us now describe the principal congruence \(\theta(a, b)\) for \(a, b \in A \in V\). Letting \(f\) and \(h\) denote algebraic functions of one variable, we recursively define

\[
A_0 = \{b\},
\]

\[
A_{i+1} = \{h(c): c \in A_i, h(a) = a, \#h < M_2\}
\]

and finally,

\[
\theta_0(a, b) = \left\{ (f(a), f(c)): c \in \bigcup_{i=1}^{\infty} A_i, \#f < M_1 \right\}.
\]

**Lemma 1.** \(\theta(a, b)\) is the transitive-symmetric hull of \(\theta_0(a, b)\).

**Proof.** It is clearly enough to see that this hull is closed under the actions of OAF's, i.e. algebraic functions \(g: A \to A\) formed by freezing all but one of the places in one of the operations \(G\) of \(A\). In fact it is obviously enough to consider arbitrary \((f(a), f(c)) \in \theta_0(a, b)\) and show that \((gf(a), gf(c))\) is in the transitive-symmetric hull of \(\theta_0(a, b)\). By definition of \(\theta_0(a, b)\), we have \(c \in A_i\) for some \(i\), and \(#f < M_1\). Now consider the algebraic functions

\[
h_j(z) = P_j(gfz, gfa, a) \quad (1 < j < 2N).
\]

Clearly \(#h_j < M_2\), and \(h_j(a) = a\), by our identities; thus each \(h_j(c) \in A_{i+1}\). Next consider the algebraic functions

\[
f_i(z) = G_i(gf(c), gfa, a, z) \quad (1 < i < N).
\]

It is clear that \(#f_i < M_1\), and thus we have \((f_i(a), f_ih_j(c)) \in \theta_0(a, b)\); in particular, the following pairs are in \(\theta_0(a, b)\):
In the identities stated above, if we now substitute $gf(c)$ for $x$, $gf(a)$ for $y$, and $a$ for $z$, we obtain

\[ gf(c) = f_1h_1c, \]
\[ f_1h_2c = f_2h_3c, \]
\[ f_2h_4c = f_3h_5c, \]
\[ \vdots \]
\[ f_nh_{2N-1}c = f_nh_{2N}c. \]

Combining these equalities with our list of pairs from $\theta_0(a, b)$, we easily obtain that $(gf(a), gf(c))$ is in the transitive-symmetric hull of $\theta_0(a, b)$.

**Lemma 2.** There exists a function $v: \omega \to \omega$ (depending only on $V$, which is assumed congruence-regular) with the following property. If $a, b, c, d \in \mathfrak{A} \in \mathfrak{V}$ and $(c, d) \in \Theta(a, b)$, then $(c, d)$ is in the transitive-symmetric hull of a set of $M$ pairs $(f(a), f(b))$, with each $f$ an algebraic function of rank $< v(M)$, where $M$ denotes the power of the smallest congruence class containing $a, b, c$ and $d$.

**Proof.** Consider the representation in Lemma 1. It is clear that $A_M = A_{M+1} = \ldots$, since a sequence $b, f_1(b), f_2f_1(b), f_3f_2f_1(b), \ldots$ with each $f_i(a) = a$, must repeat itself at least once in $M$ steps, and hence can be shortened if it is longer than $M$. This obviously places a bound on the ranks of algebraic functions required. And the length of the necessary transitive chain is obviously limited by $M$.

Our next lemma is well known for groups (see [6, 51.23, p. 146]), limiting the orders of chief factors in varieties generated by a finite group. We have heard that the general case was previously discovered by J. B. Nation.

**Lemma 3.** If $\mathfrak{A}$ is finite and $\text{HSP} \mathfrak{A}$ is congruence-permutable, then every finite $\mathfrak{B} \in \text{HSP} \mathfrak{A}$ has a strictly increasing sequence of congruence blocks $B_0 < B_1 < \cdots < B_k = \mathfrak{B}$, with $|B_i| < |\mathfrak{A}|'$ for each $i < k$.

**Proof.** Let us have $\mathfrak{B} = \mathfrak{C}/\theta$ with $\mathfrak{C} \subseteq \mathfrak{A}^m$. We define congruences $\theta_j (0 < j < m)$ on $\mathfrak{C}$ as follows:

\[ c \theta_j c' \leftrightarrow c_i = c'_i \quad (i = j + 1, j + 2, \ldots, m). \]

(And thus $\Delta = \theta_0 \subseteq \theta_1 \subseteq \cdots \subseteq \theta_m = \mathfrak{C}^2$.) Let us look at one $(\theta \vee \theta_j)$-block $U$, and count the number of $(\theta \vee \theta_{j-1})$-blocks which it contains. Fix $c \in U$. Congruence-permutability tells us that for each $y \in U$ we have $c \theta_j z$ $\theta y$ for some $z \in \mathfrak{C}$. Suppose we also have $c \theta_j z' \theta y'$ with $z$ and $z'$ agreeing in
their $j$th coordinate; then the definition of $\theta_j$ tells us that $z \theta_{j-1} z'$, and hence $y(\theta \vee \theta_{j-1})y'$. Thus the $(\theta \vee \theta_{j-1})$-block of $y \in U$ is determined by the $j$th coordinate of $z$, which can take only $|A|$ values. Hence each $(\theta \vee \theta_j)$-block contains at most $|A|$ $(\theta \vee \theta_{j-1})$-blocks. Regarding each $\theta \vee \theta_j$ as a congruence $\psi_j$ on $B = G/\theta$, we have the corresponding property: each $\psi_j$-block contains at most $|A|$ $\psi_{j-1}$-blocks.

It is now easy to construct the desired sequence of congruence blocks; we start at the top and work downward. Define $U_0 = B$ and $U_{i-1} = \text{the largest of all } \psi_j\text{-blocks (for any } j)\text{ which are proper subsets of } U_i$. Since $B$ is finite and $\psi_0 = \Delta_B$, the process terminates at some singleton $U_k$. Renumbering upward via $B_i = U_{k-i}$ finishes the construction.

In the spirit of [2] and [8] we define a congruence formula to be any positive 4-ary formula $\varphi(-, -, \cdot, \cdot)$ obeying $\forall y \exists x[\varphi(x, x, y, z) \rightarrow y = z]$. It is shown in [8] that for $a, b \in A$, the principal congruence $\theta(a, b)$ consists of all pairs $(c, d)$ for which $A \equiv \varphi(a, b, c, d)$ for some congruence formula $\varphi$. (Our proof in [8] was based on Mal'tsev's description mentioned above; a direct proof is even easier: just show that all such $(c, d)$ form a congruence relation.) We now refine this congruence-formula description for regular and permutable varieties generated by a finite algebra.

**Lemma 4.** If $\mathfrak{V} = HSP A$ (with $A$ finite) is regular and permutable, then there exists a sequence $\varphi_1, \varphi_2, \ldots$ of congruence formulas with the following property. The elements of every finite s.i. algebra $G \in \mathfrak{V}$ can be arranged in a sequence $b_0, b_1, b_2, \ldots$ such that $G \equiv \varphi_j(b_i, b_j, b_0, b_1)$ whenever $i < j$.

**Proof.** Take $B_0 \subset B_1 \subset B_2 \subset \ldots$ as provided by Lemma 3. By congruence-regularity, the minimum congruence $\theta$ on $G$ contains a pair $(b_0, b_1)$ with $b_0 b_1 \in B_1$ and $b_0 \neq b_1$. Now simply choose the sequence so that for each $j$, $b_j \in B_j$. Certainly for $i < j$ we have $(b_0, b_i) \in \theta(b_i, b_j)$. By Lemma 2, $(b_0, b_i)$ is in the transitive-symmetric hull of at most $|A|$ pairs $(f(b_i), f(b_j))$, with each $f$ an algebraic function of rank $< r(|A|)$. It should now be clear how we build the formula $\varphi_j$. We write down all "chains" linking $b_0$ with $b_i$ with $< |A|$ links composed of pairs $(f(b_i), f(b_j))$ (there are a finite number), form their disjunction, and finally existentially quantify over all parameters appearing in the terms designating algebraic functions.

Notice that the conclusion of this lemma involves a weakened version of Baldwin and Berman's notion [2] of definable principal congruences.

**Theorem.** If $A$ is a finite algebra with $HSP A$ congruence-regular and congruence-permutable, and if $HSP A$ contains arbitrarily large finite subdirectly irreducible algebras, then it contains an infinite subdirectly irreducible algebra.

**Proof.** Let $G_i$ be s.i. with $|G_i| > i$ ($i > 1$). Write $G_i = \{b_{0i}, b_{1i}, b_{2i}, \ldots\}$ according to Lemma 4. Let $G$ be a nonprincipal ultraproduct of $\langle G_i : i \in I \rangle$.
and take \( b_0, b_1, b_2, \ldots \in \mathfrak{V} \) such that for each \( j \), the \( i \)th coordinate of \( b_j \) is \( b_{ij} \) (a.e. in \( i \)). Los' theorem tells us that \( \mathfrak{V} \models \varphi_i(b_i, b_j, b_0, b_1) \) whenever \( i < j \) where \( \varphi_i \) is as in Lemma 4. Thus all \( b_j \) must be distinct in \( \mathfrak{V} / \theta \) whenever \( (b_0, b_1) \notin \theta \), and hence such a quotient \( \mathfrak{V} / \theta \) is always infinite. If we take \( \theta \) to be a maximal congruence separating \( b_0 \) and \( b_1 \), then \( \mathfrak{V} / \theta \) is infinite and s.i. □

REFERENCES


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