PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 75, Number 2, July 1979

A NOTE ON THE LOCALIZATION THEOREM FOR PROJECTIVE MODULES

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ABSTRACT. Let R be a ring and S a central multiplicative subset. An example is given to show that the localization theorem for projective modules, valid when S consists of non-zero-divisors, does not hold when S is allowed to contain zero-divisors.

Let R be a ring with 1, and S a multiplicatively closed central subset. Let H denote the (exact) category consisting of those finitely generated R-modules M admitting a resolution of length one by finitely generated projective R-modules, and such that $M_S = 0$. When S consists of non-zero-divisors, Quillen and others have established a long exact localization sequence ([1]-[3])

$$\ldots \to K_1H \to K_1R \to K_1R_S \to K_0H \to K_0R \to K_0R_S.$$

The purpose of this note is to give a class if simple examples showing that, in general, the hypothesis on S cannot be relaxed.

Let R be any commutative Noetherian local ring with dim $R \geqslant 1$, depth $R = 0$. Let f be any nonnilpotent element of R, and put $S = \{1, f, f^2, \ldots\}$. By hypothesis, all elements of the maximal ideal of R are zero-divisors. Then a simple argument ([4, Lemma 4, p. 182]) shows that any finitely generated R-module of finite projective dimension is, in fact, projective (and consequently, free). Thus $H = 0$ in this case. If an exact sequence existed in the form above, it would follow that the map $K_nR \to K_nR_S$ would be an isomorphism for $n \geqslant 1$.

Now, since R is local, $K_1(R) = R^*$, where R^* is the multiplicative group of units of R. Consider $1/f \in R_S^* \subset K_1(R_S)$; we claim that $1/f$ is not in the image of $R^* \to R_S^* \subset K_1(R_S)$. For otherwise there would exist $a \in R^*$ such that $a/1 = 1/f$ in R_S. This implies that $f^n(1 - af) = 0$ in R for some $n > 0$. But $1 - af$ is a unit, so $f^n = 0$, a contradiction.

As an example, let k be a field, and put $R = A_m$, where $A = k[x, y]/(x^2, xy)$, and m is the maximal ideal generated by the images of x and y in A; let f be the image of y in R.

Received by the editors October 28, 1978.
AMS (MOS) subject classifications (1970). Primary 18F25.

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