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# THE $p$-ADIC GAMMA MEASURES

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ABSTRACT. The $p$-adic log gamma function and its derivatives are used to define distributions and measures on the $p$-adic units. These measures are then used to interpolate the Leopoldt-Kubota $p$-adic $L$-functions on the positive integers.

**1. Introduction.** If $X$ is a compact-open subset of $Q_p$, a $p$-adic distribution on $X$ is an additive mapping from the compact open subsets of $X$ into $\Omega_p$. A bounded distribution is called a measure.

B. Mazur has used the Bernoulli polynomials $B_k$ to define an important family of distributions and measures on the $p$-adic integers. He then used the Bernoulli measures to express certain $p$-adic $L$-functions as integrals.

The Bernoulli distributions are defined for $k = 0, 1, \ldots$ by $\mu_{B,k}(a + p^m Z_p) = p^{m(k-1)} B_k(a/p^m)$ where $a,m \in Z$, $m \geqslant 0$, $0 \leqslant a < p^m$. The Bernoulli measures are defined for each $p$-adic unit $\alpha$ by $\mu_{k,\alpha}(A) = \mu_{B,k}(A) - \alpha^{-k}\mu_{B,k}(\alpha A)$.

The purpose of this article is to use the $p$-adic log gamma function to define a set of distributions and measures which complement the Bernoulli distributions and measures.

The Bernoulli distributions are unique in the sense that the only polynomials $Q(X)$ that yield an additive function when used in the formula $\mu(a + p^m Z_p) = p^{m(k-1)} Q(a/p^m)$, are the constant multiples of the Bernoulli polynomials.

The gamma distributions satisfy this uniqueness property, if, instead of polynomials, we use functions of the form $Q(x) = \sum_{n=1}^{\infty} a_n/x^n$ and define $\mu$ on the $p$-adic units.

The Bernoulli measures are all related to $\mu_{1,\alpha}$ by the property

$$\int_X 1 \cdot \mu_{k,\alpha} = \int_X kx^{k-1}\mu_{1,\alpha}.$$

This is analogous to the relation between $dx^k$ and $dx$ for $k$ a nonnegative integer. The gamma measures are the measures which relate to $\mu_{1,\alpha}$ as $dx^k$ relates to $dx$ with $k$ a negative integer.

Mazur found that the Bernoulli measures allow us to write $L_p(r, \chi)$ as an integral when $r$ is a negative integer. The integral has $r$ as a parameter, and is a continuous function of $r$ on the $p$-adic integers. Hence, by a continuity

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argument there is an integral formula for $L_p(r, \chi)$. The gamma measures allow us to calculate Mazur's integral when $r$ is a positive integer, so that we obtain the integral formula for $L_p(r, \chi)$ by interpolation across the positive integral values of $r$.

A description of Bernoulli measures and distributions can be found in Koblitz's book [4].

**2. Notation.** We will use $Q_p$, $Z$, $Z_p$, $Z_p^x$ and $\Omega_p$ for, respectively, the $p$-adic completion of the rational numbers, the ring of rational integers, the $p$-adic completion of $Z$ in $Q_p$, the units of $Z_p$ and the completion of the algebraic closure of $Q_p$. The log function is always the $p$-adic logarithm with $\log p = 0$, as described by Iwasawa in [2].

If $a,b \in \Omega_p$, we will write $a \equiv b \pmod{p^m}$ to mean $|a - b|_p \leq p^{-m}$.

**3. Gamma distributions.** In order to define an additive function $\mu$ on the compact-open subsets of $Z_p^x$ it is sufficient (see [4]) to define $\mu$ on sets of the form $a + p^m Z_p$ with $a,m$ positive integers, $(a,p) = 1$, $0 < a < p^m$ and then check that

$$\mu(a + p^m Z_p) = \sum_{b=0}^{p-1} \mu\left(a + bp^m + p^{m+1} Z_p\right).$$

$\mu$ is then extended to all compact-open sets by

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i), \qquad A_i \cap A_j = \varnothing \text{ if } i \neq j.$$

Definition. For each nonnegative integer $k$ we define $\nu_{G,k}$ by $\nu_{G,k}(a + p^m Z_p) = p^{-km} G_p^{(k)}(a/p^m)$, where $a,m \in Z$, $0 < a < p^m$, $(a,p) = 1$, and $G_p^{(k)}$ is the $k$th derivative of the $p$-adic log gamma function $G_p$ [1].

The additivity of $\nu_{G,k}$ is a consequence of the Gauss multiplication formula for $G_p^{(k)}$ [1]:

$$G_p^{(k)}(x) = p^{-k} \sum_{b=0}^{p-1} G_p^{(k)}\left(\frac{x+b}{p}\right).$$

A uniqueness theorem for the $\nu_{G,k}$ when $k \geq 2$ follows easily from

Proposition 1. *If*

$$f(x) = \sum_{n=k}^{\infty} \frac{a_n}{x^n}, \qquad a_n \in \Omega_p, \quad a_k \neq 0, \quad k \geq 1,$$

*is defined for* $|x|_p > R \geq 1$, *and*

$$f(px) = \varepsilon \sum_{b=0}^{p-1} f\left(x + \frac{b}{p}\right) \tag{1}$$

*where* $\varepsilon \in \Omega_p$ *and* $|x|_p > pR$, *then* $\varepsilon = p^{-k-1}$ *and* $f(x)$ *is a constant multiple of* $G_p^{(k+1)}(x)$.

PROOF. We have, by using the binomial expansion for $(x + b/p)^{-n}$,

$$\sum_{b=0}^{p-1} f(x + b/p) = \sum_{m=k}^{\infty} \frac{1}{p^m x^m} \sum_{n=k}^{m} a_n p^n S_{m-n} \binom{-n}{m-n}$$

where $S_0 = p$ and $S_r = \sum_{b=0}^{p-1} b^r$.

Then, applying (1) and equating coefficients, we find that $a_k = \varepsilon a_k p^{k+1}$, so $\varepsilon = p^{-k-1}$ and, for $m > k$, $a_m$ is determined by

$$a_m = (1 - p^{-m-k})^{-1} p^{-k-1} \sum_{n=k}^{m-1} a_n p^n S_{m-n} \binom{-n}{m-n}.$$

Since $G_p^{(k+1)}(x)$ satisfies (1) it is clear from the recursion relation for the $a_m$ that $f(x)$ is a constant multiple of $G_p^{(k+1)}$. $\square$

It is not difficult to show that if $f(x)$ is given by a Laurent series for $|x|_p > R \geqslant 1$, and $f(x)$ satisfies (1), then, up to a constant multiple, $f$ is either a Bernoulli polynomial $B_k(x)$ with $k \geqslant 0$ and $\varepsilon = p^{k-1}$ or $f(x) = G_p^{(k+1)}(x)$ with $k \geqslant 1$ and $\varepsilon = p^{-k-1}$.

The uniqueness result for $\nu_{G,k}$ is

THEOREM 1. *If $\nu$ is a distribution on $Z_p^x$ with values in $\Omega_p$ and $\nu(a + p^m Z_p) = \varepsilon^m f(a/p^m)$ where $a,m \in Z$, $(a,p) = 1$, $0 < a < p^m$, $\varepsilon \in \Omega_p$ and*

$$f(x) = \sum_{n=k}^{\infty} \frac{a_n}{x^n}, \qquad a_k \neq 0, \quad k \geqslant 1,$$

*then $\varepsilon = p^{-k-1}$ and $f(x)$ is a constant multiple of $G_p^{(k+1)}(x)$.*

PROOF. To satisfy the additivity condition, we need

$$\varepsilon^m f\left(\frac{a}{p^m}\right) = \nu(a + p^m Z_p) = \sum_{b=0}^{p-1} \nu(a + bp^m + p^{m+1} Z_p)$$

$$= \varepsilon^{m+1} \sum_{b=0}^{p-1} f\left(\frac{a}{p^{m+1}} + \frac{b}{p}\right).$$

Hence, $f(px) = \varepsilon \sum_{b=0}^{p-1} f(x + b/p)$ for all $x$ of the form $p^{-m}$ with $m > 1$. Since $\lim_{m\to\infty} p^{-m} = \infty$, the identity is valid for all $x$ with $|x|_p > p$.

**4. Gamma measures.** Mazur defined the Bernoulli measures $\mu_{k,\alpha}$ by the following formula. For each $\alpha \in Z_p^x$,

$$\mu_{k,\alpha}(A) = \mu_{B,k}(A) - \alpha^{-k} \mu_{B,k}(\alpha A).$$

In a similar vein, we define the gamma measures $\nu_{k,\alpha}$ as follows:

$$\nu_{0,\alpha}(A) = \nu_{G,0}(A) - \alpha^{-1} \nu_{G,0}(\alpha A) + ((\log \alpha)/\alpha)\, \mu_{B,1}(\alpha A),$$

$$\nu_{1,\alpha}(A) = \nu_{G,1}(A) - \nu_{G,1}(\alpha A) + (\log \alpha)\, \mu_{B,0}(A),$$

$$\nu_{k,\alpha}(A) = ((-1)^k / (k-2)!)(\nu_{G,k}(A) - \alpha^{k-1} \nu_{G,k}(\alpha A)) \quad \text{for } k = 2, 3, \ldots.$$

It is clear from the definitions that each $\nu_{k,\alpha}$ is a distribution. The fact that the values of each $\nu_{k,\alpha}$ are bounded is implied by the following theorem.

THEOREM 2. *If* $\alpha \in Z_p^x$, $a,m \in Z$, $(a,p) = 1$ *and* $0 < a < p^m$, *then*

(i) $\nu_{0,\alpha}(a + p^mZ_p) \equiv (\log a)\,\mu_{1,\alpha}(a + p^mZ_p) \pmod{p^m}$,

(ii) *if* $p > 3$, $\nu_{1,\alpha}(a + p^mZ_p) \equiv (1/a)\,\mu_{1,\alpha}(a + p^mZ_p) \pmod{p^m}$, *if* $p = 2$, *use* mod $p^{m-2}$ *and if* $p = 3$, *use* mod $p^{m-1}$,

(iii) *if* $p > 3$ *and* $k \geqslant 2$, $\nu_{k,\alpha}(a + p^mZ_p) \equiv (1-k)a^{-k}\mu_{1,\alpha}(a + p^mZ_p) \pmod{p^m}$.

*If* $p = 2, 3$, *use* mod $p^{m-1}$ *instead of* mod $p^m$.

PROOF. We will show that, if $p > 3$,

$$\nu_{1,\alpha}(a + p^mZ_p) \equiv (1/a)\,\mu_{1,\alpha}(a + p^mZ_p) \pmod{p^m}.$$

The other results are proved in a similar manner.

For $x \in Z_p^x$ we define $\{x\}_m$ to be the unique positive integer $<p^m$ satisfying $\{x\}_m \equiv x \pmod{p^m}$, then $[x/p^m]$ is defined by

$$\left[\frac{x}{p^m}\right] = \frac{x - \{x\}_m}{p^m}.$$

If $x \in Z$, then [ ] is just the usual "greatest integer" symbol.

From the definition we have

$$\nu_{1,\alpha}(a + p^mZ_p) = p^{-m}G_p'\left(\frac{a}{p^m}\right) - p^{-m}G_p'\left(\frac{\{a\alpha\}_m}{p^m}\right) + \frac{\log\alpha}{p^m}.$$

Now we use the Stirling series [1] valid for $|x|_p > 1$,

$$G_p(x) = \left(x - \frac{1}{2}\right)\log(x) - x + \sum_{r=1}^{\infty}\frac{B_{r+1}}{r(r+1)x^r},$$

to obtain

$$\begin{aligned}\nu_{1,\alpha}(a + p^mZ_p) &\equiv p^{-m}\left(\log\left(\frac{a\alpha}{\{a\alpha\}_m}\right) - \frac{p^m}{2a} + \frac{p^m}{2\{a\alpha\}_m}\right) \pmod{p^m}\\ &\equiv p^{-m}\log\left(1 + \frac{[a\alpha/p^m]p^m}{\{a\alpha\}_m}\right) + \frac{1}{2}\left(\frac{1}{a}\right)\left(\frac{1}{\alpha} - 1\right) \pmod{p^m}\\ &\equiv \frac{1}{a}\left(\frac{1}{\alpha}\left[\frac{a\alpha}{p^m}\right] + \frac{1}{2}\left(\frac{1}{\alpha} - 1\right)\right) \pmod{p^m}\end{aligned}$$

which, by definition of $\mu_{1,\alpha}$, (see [4])

$$= (1/a)\,\mu_{1,\alpha}(a + p^mZ_p).$$

Theorem 2 not only shows each $\nu_{k,\alpha}$ is bounded, and therefore a measure, but also that $\nu_{1-k,\alpha}$ relates to $\mu_{1,\alpha}$ as $dx^k$ relates to $dx$ when $k$ is a negative integer. Furthermore, using log to mean real log, we can say that $\nu_{0,\alpha}$ is analogous to $d(x(\log x) - x)$ and $\nu_{1,\alpha}$ is analogous to $d(\log x)$.

The analogies can be made precise with a corollary to Theorem 2.

COROLLARY. *If $A$ is a compact-open subset of $Z_p^x$ and $f$: $A \to \Omega_p$ is continuous, then*

(i) $\int_A f(x)\nu_{0,\alpha} = \int_A f(x)(\log x)\,\mu_{1,\alpha}$,

(ii) $\int_A f(x)\nu_{1,\alpha} = \int_A (f(x)/x)\,\mu_{1,\alpha}$,

(iii) *if* $k \geqslant 2$, $\int_A f(x)\nu_{k,\alpha} = \int_A f(x)(1-k)x^{-k}\mu_{1,\alpha}$.

PROOF. By definition [4],

$$\int_A f(x)\nu_{k,\alpha} = \lim_{m\to\infty} \sum_{\substack{a=0 \\ a+p^mZ_p \subset A}}^{p^m-1} f(a)\nu_{k,\alpha}(a+p^mZ_p).$$

The application of the congruences in Theorem 2 produces the corollary.

The following property of $\nu_{k,\alpha}$ is a direct consequence of the extension theorem for $G_p$ [1]: $G_p(x) + G_p(1-x) = 0$.

THEOREM 3. $\nu_{k,\alpha} = \nu_{k,-\alpha}$ *for* $k = 0, 1, \ldots.$

**5. Applications.** We begin with the calculation of an integral.

PROPOSITION 2. *If $a,m,k \in Z$, $(a,p) = 1$, $0 < a < p^m$ and $k \geqslant 2$, then*

$$\int_{a+p^mZ_p} t^{-k}\mu_{1,\alpha}(t) = \frac{(-1)^k p^{-mk}}{(k-1)!}\left(\alpha^{k-1}G_p^{(k)}\left(\frac{\{a\alpha\}_m}{p^m}\right) - G_p^{(k)}\left(\frac{a}{p^m}\right)\right).$$

PROOF.

$$\begin{aligned}\int_{a+p^mZ_p} t^{-k}\mu_{1,\alpha}(t) &= \frac{1}{1-k}\int_{a+p^mZ_p} 1\cdot\nu_{k,\alpha} = \frac{1}{1-k}\nu_{k,\alpha}(a+p^mZ_p)\\ &= \frac{(-1)^k p^{-mk}}{(k-1)!}\left(\alpha^{k-1}G_p^{(k)}\left(\frac{\{a\alpha\}_m}{p^m}\right) - G_p^{(k)}\left(\frac{a}{p^m}\right)\right).\end{aligned}$$

We can use Proposition 2 to obtain a formula for $G_p^{(k)}(a/p^m)$.
This formula has been found with a different technique by Koblitz [3].

THEOREM 4. *If $a,m,k \in Z$, $p \nmid a$, $0 < a < p^m$, $k \geqslant 2$, $\alpha \neq 1$, $\alpha \in Z_p^x$ and $\alpha \equiv 1 \pmod{p^m}$, then*

$$G_p^{(k)}\left(\frac{a}{p^m}\right) = \frac{(-1)^k p^{mk}(k-1)!}{\alpha^{k-1}-1}\int_{a+p^mZ_p} t^{-k}\mu_{1,\alpha}(t).$$

PROOF. Combine Proposition 2 with the observation that $\alpha \equiv 1 \pmod{p^m}$ implies $\{a\alpha\}_m = a$.

We have a similar result for the values of $G_p(a/p^m)$. The proof is essentially the same as that of Theorem 4.

THEOREM 5. *If* $a,m \in Z$, $(a,p) = 1$, $0 < a < p^m$, $\alpha \in Z_p^\times$, $\alpha \neq 1$ *and* $\alpha \equiv 1 \pmod{p^m}$, *then*

$$G_p\left(\frac{a}{p^m}\right) = \frac{\log \alpha}{1-\alpha}\left(\frac{a\alpha}{p^m} - \frac{1}{2}\right) + \frac{\alpha}{\alpha - 1}\int_{a+p^mZ_p} (\log t)\, \mu_{1,\alpha}(t).$$

Since $\nu_{k,\alpha} = \nu_{k,-\alpha}$ (Theorem 3), there are formulas for $G_p(a/p^m)$ and $G_p^{(k)}(a/p^m)$ when $\alpha$ is near $-1$. Namely:

THEOREM 6. *If* $\alpha \equiv -1 \pmod{p^m}$ *and* $a,m,k$ *are as in Theorems* 4 *and* 5, *then*

$$G_p\left(\frac{a}{p^m}\right) = \frac{-\log \alpha}{1+\alpha}\left(\frac{a\alpha}{p^m} + \frac{1}{2}\right) + \frac{\alpha}{\alpha + 1}\int_{a+p^mZ_p} (\log t)\, \mu_{1,\alpha}(t),$$

$$G_p^{(k)}\left(\frac{a}{p^m}\right) = \frac{(-1)^k p^{mk}(k-1)!}{(-\alpha)^{k-1} - 1}\int_{a+p^mZ_p} t^{-k}\mu_{1,\alpha}(t).$$

$L$-FUNCTIONS. The integral in Proposition 2 can be used to do an interpolation of $L_p(r, \chi)$ on the positive integral values of $r$. We then obtain Mazur's formula for $L_p(r, \chi)$ when $\chi$ is a Dirichlet character mod $p^m$.

For $x \in Z_p^\times$, let $\omega(x)$ be the unique $(p-1)$th root of unity $\equiv x \pmod p$. Define $\langle x \rangle$ as $x\omega^{-1}(x)$. If $\chi$ is a Dirichlet character mod $p^m$, extend it to a mapping on $Z_p^\times$ by letting $\chi(x) = \chi(\{x\}_m)$.

THEOREM 7 (MAZUR). *If $\chi$ is a Dirichlet character* mod $p^m$, $\alpha \in Z_p^\times$, *and* $|r|_p \leqslant 1$, *then*

$$\left(\langle\alpha\rangle^{r-1}\bar{\chi}(\alpha) - 1\right)L_p(r, \chi) = \int_{Z_p^\times} \frac{\chi(t)}{t\langle t\rangle^{r-1}}\, \mu_{1,\alpha}(t).$$

*The left-hand factor on the left side is zero just when either $\alpha$ is a root of unity with $\chi(\alpha) = 1$ or when $r = 1$ and $\chi(\alpha) = 1$.*

PROOF. We will use $\Sigma^*$ to indicate a sum in which the index of summation takes on only values not divisible by $p$.

If $r \in Z$, $r \geqslant 2$, then

$$\begin{aligned}
\int_{Z_p^\times} \frac{\chi(t)}{t\langle t\rangle^{r-1}}\, \mu_{1,\alpha}(t) &= \sum_{a=1}^{p^m}{}^{*}\, \chi(a)\omega^{r-1}(a)\int_{a+p^mZ_p} t^{-r}\mu_{1,\alpha}(t) \\
&= \sum_{a=1}^{p^m}{}^{*}\, \frac{\chi(a)\omega^{r-1}(a)(-1)^r p^{-mr}}{(r-1)!}\left(\alpha^{r-1}G_p^{(r)}\left(\frac{\{a\alpha\}_m}{p^m}\right) - G_p^{(r)}\left(\frac{a}{p^m}\right)\right) \\
&= \left(\langle\alpha\rangle^{r-1}\bar{\chi}(\alpha) - 1\right)\frac{(-1)^r p^{-mr}}{(r-1)!}\sum_{a=1}^{p^m}{}^{*}\, \chi(a)\omega^{r-1}(a)G_p^{(r)}\left(\frac{a}{p^m}\right) \\
&= \left(\langle\alpha\rangle^{r-1}\bar{\chi}(\alpha) - 1\right)L_p(r, \chi).
\end{aligned}$$

This last step followed from comparing the definitions of $G_p^{(r)}(x)$ at $x = ap^{-m}$ [1] and $L_p(r, \chi)$ [5]:

$$G_p^{(r)}(x) = (-1)^r(r-2)! \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \frac{1}{(x+n)^{r-1}}$$

$$L_p(r, \chi) = \frac{1}{r-1} \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=1}^{p^k}{}^{*} \frac{\chi(n)}{\langle n\rangle^{r-1}} \qquad \text{if } \chi \text{ is } (\operatorname{mod} p^m).$$

Since each side of the equation in Theorem 7 is continuous on $Z_p$, and the integers $\geqslant 2$ are dense in $Z_p$, the equality is valid for all $r \in Z_p$.

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