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SOME INEQUALITIES OF ALGEBRAIC POLYNOMIALS HAVING REAL ZEROS

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Dedicated to Professor A. Zygmund

ABSTRACT. Let $P_n(x)$ be an algebraic polynomial of degree n having all real zeros. We set

$$I_n = \frac{\|P_n'(x)\omega(x)\|_{L_2[a,b]}}{\|P_n(x)\omega(x)\|_{L_2[a,b]}}.$$

In this work the lower and upper bounds of I_n are investigated under the assumptions that all the zeros of $P_n(x)$ are inside $[a, b]$ and outside $[a, b]$, respectively. We restrict ourselves here with two cases, (1) $\omega(x) = (1 - x^2)^{1/2}$, $[a, b] = [-1, 1]$; (2) $\omega(x) = e^{-x/2}$, $[a, b] = [0, \infty)$. Results are shown to be best possible.

In this work we are concerned with the following theorems of P. Turán [3], P. Erdős [1], and A. K. Varma [5].

THEOREM A. *Let $P_n(x)$ be an algebraic polynomial of degree n having all its zeros inside $[-1, +1]$; then we have*

$$\max_{-1 \le x \le +1} |P_n'(x)| > \frac{n^{1/2}}{6} \max_{-1 \le x \le +1} |P_n(x)|. \tag{1.1}$$

($P_n'(x)$ stands for the derivative of $P_n(x)$.)

THEOREM B. *Let $P_n(x)$ be a polynomial of degree n satisfying the inequality $|P_n(x)| \le 1$ for $-1 \le x \le +1$. Suppose $P_n(x)$ has only real roots and no root inside $[-1, +1]$; then for $-1 \le x \le +1$, $|P_n'(x)| < \frac{1}{2}en$. This is the best possible result.*

THEOREM C. *Let $P_n(x)$ be an algebraic polynomial of degree n having all zeros real and inside $[-1, +1]$; then we have*

$$\|P_n'\|^2_{L_2[-1,+1]} > \left(\frac{n}{2} + \frac{3}{4} + \frac{3}{4n}\right)\|P_n\|^2_{L_2[-1,+1]}, \qquad n \ge 13, \tag{1.2}$$

where

$$\|P_n\|^2_{L_2[-1,+1]} = \int_{-1}^{1} P_n^2(x)dx.$$

Further, if $P_n(+1) = P_n(-1) = 0$ and $n \ge 2$ then under the above conditions

$$\|P_n'\|^2_{L_2[-1,+1]} \ge \left(\frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)}\right)\|P_n\|^2_{L_2[-1,+1]}$$

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with equality holding for $P_n(x) = (1 - x^2)^m$, $n = 2m$.

The above Theorem C is analogous to Theorem A in the L_∞ norm. In view of the above theorems it is natural to ask about lower bound (upper bound) of the expression

$$\|\omega(x)P_n'(x)\|_{L_2[a,b]}/\|\omega(x)P_n(x)\|_{L_2[a,b]}$$

under the assumption that all the zeros of $P_n(x)$ are all inside $[a, b]$ (no roots inside $[a, b]$, respectively). We limit ourselves here in this work to the above question when $[a, b] = [0, \infty)$, $\omega(x) = e^{-x/2}$ and $[a, b] = [-1, +1]$, $\omega(x) = (1 - x^2)^{1/2}$.

Throughout x_k's denote the zeros of $P_n(x)$. Here we prove the following theorems.

THEOREM 1. *Let* $P_n(x)$ *be an algebraic polynomial of degree* n *having all zeros inside* $[0, \infty)$. *Let* $P_n(0) = 0$ *or*

$$\sum_{k=1}^{n} \frac{1}{x_k} \geqslant \frac{1}{2};$$

then

$$\|e^{-x/2}P_n'\|^2_{L_2[0,\infty)} \geqslant \frac{n}{2(2n-1)}\|e^{-x/2}P_n\|^2_{L_2[0,\infty)} \tag{1.3}$$

with equality for $P_n(x) = x^n$.

The following remark is needed concerning the above theorem. Let us choose $P_n(x) = (x - x_1)^n$, $x_1 > 0$. It can be shown easily that the ratio of

$$\int_0^\infty (P_n'(x))^2 e^{-x}dx \Big/ \int_0^\infty (P_n(x))^2 e^{-x}dx$$

can be made arbitrary small for such polynomials by choosing x_1 arbitrary large. Thus some condition on the roots such as $\sum_{k=1}^n (x_k)^{-1} \geqslant \frac{1}{2}$ is natural.

Next, we turn to the case when $\omega(x) = (1 - x^2)^{1/2}$, $[a, b] = [-1, +1]$. Here we prove

THEOREM 2. *Let* $P_n(x)$ *be an algebraic polynomial of degree* n *having all zeros real and inside* $[-1, +1]$. *Then for* $n \geqslant 2$ *we have*

$$\|(1 - x^2)^{1/2}P_n'\|^2_{L_2[-1,+1]} \geqslant \left(\frac{n}{2} + \frac{1}{4} - \frac{1}{4(n+1)}\right)\|(1 - x^2)^{1/2}P_n\|^2_{L_2[-1,+1]} \tag{1.4}$$

with equality for $P_n(x) = (1 - x^2)^m$, $n = 2m$.

The next two theorems are analogous to Theorem B of P. Erdős [1].

THEOREM 3. *Let $P_n(x)$ be an algebraic polynomial of degree $< n$ having all real roots and no root inside the interval $[-1, +1]$; then we have*

$$\left\|(1-x^2)^{1/2}P_n'(x)\right\|^2_{L_2[-1,+1]} < \frac{n(n+1)(2n+3)}{4(2n+1)}\left\|(1-x^2)^{1/2}P_n(x)\right\|^2_{L_2[-1,+1]} \tag{1.5}$$

with equality for $P_n(x) = (1+x)^n$ or $P_n(x) = (1-x)^n$.

THEOREM 4. *Let $x_1, x_2, \ldots, x_n$ be real zeros of an algebraic polynomial $P_n(x)$ of degree $< n$. Further, $-\infty < x_i < 0$, $i = 1, 2, \ldots, n$. Then*

$$\int_0^\infty xe^{-x}(P_n'(x))^2\,dx < \frac{n}{2(2n+1)}\int_0^\infty xe^{-x}(P_n(x))^2\,dx$$

with equality for $P_n(x) = x^n$.

2. Proof of Theorem 1. For the proof of this theorem we will need the following facts:

$$P_n'(x) = P_n(x)\sum_{k=1}^n \frac{1}{x-x_k}, \tag{2.1}$$

$$P_n'^2(x) - P_n(x)P_n''(x) = P_n^2(x)\sum_{k=1}^n \frac{1}{(x-x_k)^2}, \tag{2.2}$$

$$\frac{(x-x_1)^2}{(x-x_k)^2} = -1 + \frac{2(x-x_1)}{x-x_k} + \frac{(x_1-x_k)^2}{(x-x_k)^2}, \tag{2.3}$$

$$P_n'(x) - n\frac{P_n(x)}{x-x_1} = \frac{P_n(x)}{x-x_1}\sum_{k=1}^n \frac{x_k-x_1}{x-x_k}. \tag{2.4}$$

From (2.4) and the Cauchy-Schwarz inequality we have

$$\left(P_n'(x) - n\frac{P_n(x)}{x-x_1}\right)^2 < n\frac{P_n^2(x)}{(x-x_1)^2}\sum_{k=1}^n \frac{(x_k-x_1)^2}{(x-x_k)^2}. \tag{2.5}$$

From (2.5) we immediately have

$$\int_0^\infty\left(P_n'(x) - n\frac{P_n(x)}{x-x_1}\right)^2 e^{-x}\,dx < n\int_0^\infty \frac{P_n^2(x)}{(x-x_1)^2}\sum_{k=1}^n \frac{(x_k-x_1)^2}{(x-x_k)^2}e^{-x}\,dx. \tag{2.6}$$

Also, integrating by parts, we have

$$\int_0^\infty e^{-x}P_n(x)P_n'(x)dx = -\frac{1}{2}P_n^2(0) + \frac{1}{2}\int_0^\infty e^{-x}P_n^2(x)\,dx. \tag{2.7}$$

Now, we turn to the proof of Theorem 1. On integrating by parts, we have

$$\int_0^\infty e^{-x}P_n'^2(x)dx = -P_n(0)P_n'(0) - \int_0^\infty P_n(x)(P_n''(x) - P_n'(x))e^{-x}\,dx, \tag{2.8}$$

and an equivalent form

$$2\int_0^\infty e^{-x}P_n'^2(x)dx = P_n^2(0)\sum_{k=1}^n \frac{1}{x_k} + \int_0^\infty e^{-x}\big(P_n'^2(x) - P_n(x)P_n''(x)\big)dx$$
$$+ \int_0^\infty e^{-x}P_n(x)P_n'(x)dx. \tag{2.9}$$

On using (2.2) and (2.7) we obtain

$$2\int_0^\infty e^{-x}P_n'^2(x)dx = P_n^2(0)\left(-\frac{1}{2} + \sum_{k=1}^n \frac{1}{x_k}\right) + \frac{1}{2}\int_0^\infty e^{-x}P_n^2(x)dx$$
$$+ \int_0^\infty e^{-x}P_n^2(x)\sum_{k=1}^n \frac{1}{(x-x_k)^2}\,dx. \tag{2.10}$$

Now, let us denote by x_1 the first nonnegative zero of $P_n(x)$. Then we can write

$$P_n(x) = (x - x_1)q_{n-1}(x), \qquad x_1 \geqslant 0, \tag{2.11}$$

where $q_{n-1}(x)$ is a polynomial of degree $\leqslant n-1$ having real zeros lying inside $[0, \infty)$.

We set

$$I_1 = \int_0^\infty e^{-x}P_n^2(x)\sum_{k=1}^n \frac{1}{(x-x_k)^2}dx = \int_0^\infty e^{-x}q_{n-1}^2(x)\sum_{k=1}^n \frac{(x-x_1)^2}{(x-x_k)^2}dx, \tag{2.12}$$

and on using (2.3) we have

$$I_1 = -n\int_0^\infty e^{-x}q_{n-1}^2(x)dx + 2\int_0^\infty (x-x_1)q_{n-1}^2(x)\sum_{k=1}^n \frac{1}{(x-x_k)}e^{-x}dx$$
$$+ \int_0^\infty e^{-x}q_{n-1}^2(x)\sum_{k=1}^n \frac{(x_1-x_k)^2}{(x-x_k)^2}dx$$
$$= -n\int_0^\infty e^{-x}q_{n-1}^2(x)dx + 2\int_0^\infty P_n'(x)q_{n-1}(x)e^{-x}dx$$
$$+ \int_0^\infty e^{-x}q_{n-1}^2(x)\sum_{k=1}^n \frac{(x_1-x_k)^2}{(x-x_k)^2}dx,$$

$$I_1 = \frac{1}{n}\int_0^\infty P_n'^2(x)e^{-x}dx - \frac{1}{n}\int_0^\infty (P_n'(x) - nq_{n-1}(x))^2e^{-x}dx$$
$$+ \int_0^\infty e^{-x}q_{n-1}^2(x)\sum_{k=1}^n \frac{(x_1-x_k)^2}{(x-x_k)^2}dx; \tag{2.13}$$

on putting the values of I_1 from (2.13) into (2.10) we obtain

$$\left(2-\frac{1}{n}\right)\int_0^\infty e^{-x}P_n'^2(x)dx = P_n^2(0)\left(-\frac{1}{2}+\sum_{k=1}^n \frac{1}{x_k}\right)+\frac{1}{2}\int_0^\infty e^{-x}P_n^2(x)dx$$
$$+\int_0^\infty e^{-x}q_{n-1}^2(x)\sum_{k=1}^n \frac{(x_1-x_k)^2}{(x-x_k)^2}dx$$
$$-\frac{1}{n}\int_0^\infty (P_n'(x)-nq_{n-1}(x))^2e^{-x}dx. \qquad (2.14)$$

Now if $P_n(0)=0$ or $\Sigma_{k=1}^n(x_k)^{-1} \geqslant \frac{1}{2}$ then on using (2.6) we immediately obtain

$$\left(2-\frac{1}{n}\right)\int_0^\infty e^{-x}P_n'^2(x)dx \geqslant \frac{1}{2}\int_0^\infty e^{-x}P_n^2(x)dx,$$

from which the proof of Theorem 1 is complete. We would like to remark that the ratio of

$$\int_0^\infty e^{-x}P_n'^2(x)dx \Big/ \int_0^\infty e^{-x}P_n^2(x)dx$$

does indeed get smaller than $n/2(2n-1)$ for polynomials $P_0(x)=(x-x_1)^n$ for $x_1 > 2n$. For such polynomials the last two terms in (2.14) are zero and $-\frac{1}{2}+\Sigma_{k=1}^n(x_k)^{-1} < 0$.

3. Proof of Theorem 2. In order to prove Theorem 2, we need some results proved in [5]. Let $P_n(x)$ be any polynomial of degree n; then

$$2\int_{-1}^1 (1-x^2)P_n'^2(x)dx = n\int_{-1}^1 P_n^2(x)dx + S_0, \qquad (3.1)$$

$$(n+1)(2n+3)\int_{-1}^1 x^2P_n^2(x)dx = S_2 - S_0 + (n+1)\int_{-1}^1 P_n^2(x)dx$$
$$+2\int_{-1}^1 ((1-x^2)P_n'(x)+nxP_n(x))^2dx \qquad (3.2)$$

where

$$S_{2i} = \int_{-1}^1 x^{2i}P_n^2(x)\sum_{k=1}^n \frac{(1-x_k^2)}{(x-x_k)^2}dx, \qquad i=0,1,\ldots. \qquad (3.3)$$

For the proof of (3.1), (3.2) see (2.1) and (2.10) (with $r=1$) in [5]. Equation (3.2) can be rewritten in the form

$$(n+1)(2n+3)\int_{-1}^1 (1-x^2)P_n^2(x)dx$$
$$= S_0 - S_2 + 2(n+1)^2\int_{-1}^1 P_n^2(x)dx$$
$$-2\int_{-1}^1 ((1-x^2)P_n'(x)+nxP_n(x))^2dx.$$

Therefore

$$(n+1)(2n+3)\int_{-1}^{1}(1-x^2)P_n^2(x)dx < S_0 - S_2 + 2(n+1)^2\int_{-1}^{1}P_n^2(x)dx. \tag{3.4}$$

From (3.1) and (3.4) we obtain

$$\frac{2\int_{-1}^{1}(1-x^2)P_n'^2(x)\,dx}{(n+1)(2n+3)\int_{-1}^{1}(1-x^2)P_n^2(x)\,dx} > \frac{n\int_{-1}^{1}P_n^2(x)\,dx + S_0}{S_0 - S_2 + 2(n+1)^2\int_{-1}^{1}P_n^2(x)\,dx}. \tag{3.5}$$

It is given that $|x_k| < 1$, $S_0 > 0$, $S_2 > 0$ and $S_0 - S_2 > 0$ (here we use (3.3)). Hence it can be verified immediately that

$$\left(S_0 + n\int_{-1}^{1}P_n^2(x)dx\right)\Big/\left(S_0 - S_2 + 2(n+1)^2\int_{-1}^{1}P_n^2(x)dx\right) > n\big/2(n+1)^2. \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$\int_{-1}^{1}(1-x^2)P_n'^2(x)dx > \frac{n(2n+3)}{4(n+1)}\int_{-1}^{1}(1-x^2)P_n^2(x)dx.$$

This proves Theorem 2 as well.

4. Proof of Theorem 3. For the proof of Theorem 3 we need the following lemma.

LEMMA 4.1. *Let $P_n(x)$ be an algebraic polynomial of degree $< n$ having real roots and no root in the interval $[-1, +1]$. Then we have*

$$\int_{-1}^{1}(1-x^2)P_n^2(x)dx > \frac{2(2n+1)}{(n+1)(2n+3)}\int_{-1}^{1}P_n^2(x)dx. \tag{4.1}$$

PROOF It follows from the well-known theorem [2] that for such $P_n(x)$ we have

$$P_n^2(x) = \sum_{p+q=2n} a_{pq}(1+x)^p(1-x)^q, \qquad a_{pq} > 0. \tag{4.2}$$

Hence we have

$$\int_{-1}^{1}(1-x^2)P_n^2(x)dx = \sum_{p+q=2n} a_{pq}\int_{-1}^{1}(1+x)^{p+1}(1-x)^{q+1}dx.$$

But

$$\int_{-1}^{1}(1+x)^{p+1}(1-x)^{q+1}dx$$
$$= \frac{4(p+1)(q+1)}{(p+q+3)(p+q+2)}\int_{-1}^{1}(1+x)^{p}(1-x)^{q}dx.$$

Hence

$$\int_{-1}^{1}(1-x^2)P_n^2(x)dx$$
$$= 4\sum_{p+q=2n}\frac{a_{pq}(p+1)(q+1)}{(p+q+3)(p+q+2)}\int_{-1}^{1}(1+x)^{p}(1-x)^{q}dx$$
$$> \frac{4(2n+1)}{(2n+3)(2n+2)}\sum_{p+q=2n}q_{pq}\int_{-1}^{1}(1+x)^{p}(1-x)^{q}dx$$
$$= \frac{2(2n+1)}{(n+1)(2n+3)}\int_{-1}^{1}P_n^2(x)dx.$$

This proves the lemma. Now we turn to the proof of the theorem. Since $|x_k| > 1$, $k = 1, 2, \ldots, n$, it follows from (3.1) and (3.3) that

$$2\int_{-1}^{1}(1-x^2)P_n'^2(x)dx < n\int_{-1}^{1}P_n^2(x)dx, \tag{4.3}$$

with equality holding for $P_n(x) = (1+x)^p(1-x)^q$, $p+q=n$. Now, on using (4.1) and (4.3) we obtain

$$\frac{\int_{-1}^{1}(1-x^2)P_n'^2(x)dx}{\int_{-1}^{1}(1-x^2)P_n^2(x)dx} = \frac{\int_{-1}^{1}(1-x^2)P_n'^2(x)dx}{\int_{-1}^{1}P_n^2(x)dx}\,\frac{\int_{-1}^{1}P_n^2(x)dx}{\int_{-1}^{1}(1-x^2)P_n^2(x)dx}$$
$$< \frac{n}{2}\cdot\frac{(n+1)(2n+3)}{2(2n+1)}.$$

This completes the proof of Theorem 3.

5. Proof of Theorem 4. On integrating by parts

$$\int_0^\infty P_n''(x)P_n(x)e^{-x}xdx = -\int_0^\infty P_n'(x)\{P_n'(x)x + P_n(x) - xP_n(x)\}e^{-x}dx$$
$$= -\int_0^\infty xe^{-x}(P_n'(x))^2dx + \tfrac{1}{2}P_n^2(0)$$
$$+\tfrac{1}{2}\int_0^\infty xe^{-x}P_n^2(x)dx - \int_0^\infty e^{-x}P_n^2(x)dx. \tag{5.1}$$

Since $x_k < 0$, $k = 1, 2, \ldots, n$, we have

$$\int_0^\infty xe^{-x}P_n^2(x)\sum_{k=1}^n \frac{1}{(x-x_k)^2}dx < \int_0^\infty e^{-x}P_n^2(x)\sum_{k=1}^n \frac{1}{x-x_k}dx$$

$$= \int_0^\infty e^{-x}P_n(x)P_n'(x)dx$$

$$= -\tfrac{1}{2}P_n^2(0) + \frac{1}{2}\int_0^\infty P_n^2(x)e^{-x}dx. \qquad (5.2)$$

On using (5.1) and (5.2) and the identity

$$P_n'^2(x) = P_n(x)P_n''(x) + P_n^2(x)\sum_{k=1}^n \frac{1}{(x-x_k)^2}$$

we obtain

$$2\int_0^\infty P_n'^2(x)xe^{-x}dx < \frac{1}{2}\int_0^\infty xe^{-x}P_n^2(x)dx - \frac{1}{2}\int_0^\infty e^{-x}P_n^2(x)dx. \quad (5.3)$$

It is easy to see that under the conditions of our theorem

$$P_n^2(x) = \sum_{k=0}^{2n} a_k x^k, \qquad a_k > 0.$$

Hence

$$\int_0^\infty e^{-x}P_n^2(x)dx = \sum_{k=0}^{2n} a_k\int_0^\infty e^{-x}x^k dx = \sum_{k=0}^{2n} a_k k!,$$

similarly

$$\int_0^\infty e^{-x}xP_n^2(x)dx = \sum_{k=0}^{2n} a_k(k+1)!$$

$$< (2n+1)\sum_{k=0}^{2n} a_k k! = (2n+1)\int_0^\infty e^{-x}P_n^2(x)dx.$$

On using (5.3) and the above result it follows that

$$2\int_0^\infty (P_n'(x))^2 xe^{-x}dx < \left(\frac{1}{2} - \frac{1}{(2n+1)2}\right)\int_0^\infty P_n^2(x)xe^{-x}dx.$$

From this Theorem 4 now follows.

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