SOME INEQUALITIES OF ALGEBRAIC POLYNOMIALS
HAVING REAL ZEROS

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Dedicated to Professor A. Zygmund

Abstract. Let $P_n(x)$ be an algebraic polynomial of degree $n$ having all real zeros. We set

$$I_n = \frac{\|P'_n(x)\omega(x)\|_{L^1[a,b]}}{\|P_n(x)\omega(x)\|_{L^1[a,b]}}.$$

In this work the lower and upper bounds of $I_n$ are investigated under the assumptions that all the zeros of $P_n(x)$ are inside $[a, b]$ and outside $[a, b]$, respectively. We restrict ourselves here with two cases, (1) $\omega(x) = (1 - x^2)^{1/2}$, $[a, b] = [-1, 1]$; (2) $\omega(x) = e^{-x^2}$, $[a, b] = [0, \infty)$. Results are shown to be best possible.

In this work we are concerned with the following theorems of P. Turán [3], P. Erdős [1], and A. K. Varma [5].

**Theorem A.** Let $P_n(x)$ be an algebraic polynomial of degree $n$ having all its zeros inside $[-1, +1]$; then we have

$$\max_{-1 < x < +1} |P'_n(x)| \geq \frac{n^{1/2}}{6} \max_{-1 < x < +1} |P_n(x)|. \quad (1.1)$$

($P'_n(x)$ stands for the derivative of $P_n(x)$.)

**Theorem B.** Let $P_n(x)$ be a polynomial of degree $n$ satisfying the inequality $|P_n(x)| < 1$ for $-1 < x < +1$. Suppose $P_n(x)$ has only real roots and no root inside $[-1, +1]$; then for $-1 < x < +1$, $|P_n(x)| < \frac{1}{2} e^n$. This is the best possible result.

**Theorem C.** Let $P_n(x)$ be an algebraic polynomial of degree $n$ having all zeros real and inside $[-1, +1]$; then we have

$$\|P'_n\|_{L^2[-1,+1]}^2 > \left( \frac{n}{2} + \frac{3}{4} + \frac{3}{4n} \right) \|P_n\|_{L^2[-1,+1]}^2, \quad n > 13, \quad (1.2)$$

where

$$\|P_n\|_{L^2[-1,+1]}^2 = \int_{-1}^{+1} P_n^2(x) \, dx.$$

Further, if $P_n(+1) = P_n(-1) = 0$ and $n \geq 2$ then under the above conditions

$$\|P'_n\|_{L^2[-1,+1]}^2 > \left( \frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)} \right) \|P_n\|_{L^2[-1,+1]}^2$$

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with equality holding for \( P_n(x) = (1 - x^2)^m, n = 2m \).

The above Theorem C is analogous to Theorem A in the \( L_{\infty} \) norm. In view of the above theorems it is natural to ask about lower bound (upper bound) of the expression

\[
\left\| \omega(x) P_n'(x) \right\|_{L_d[a, b]} / \left\| \omega(x) P_n(x) \right\|_{L_d[a, b]}
\]

under the assumption that all the zeros of \( P_n(x) \) are all inside \([a, b]\) (no roots inside \([a, b]\), respectively). We limit ourselves here in this work to the above question when \([a, b] = [0, \infty), \omega(x) = e^{-x/2}\) and \([a, b] = [-1, +1], \omega(x) = (1 - x^2)^{1/2}\).

Throughout \( x_k \)'s denote the zeros of \( P_n(x) \). Here we prove the following theorems.

**Theorem 1.** Let \( P_n(x) \) be an algebraic polynomial of degree \( n \) having all zeros inside \([0, \infty)\). Let \( P_n(0) = 0 \) or

\[
\sum_{k=1}^{n} \frac{1}{x_k} > \frac{1}{2};
\]

then

\[
\left\| e^{-x/2} P_n' \right\|_{L_d(0, \infty)}^2 > \frac{n}{2(2n - 1)} \left\| e^{-x/2} P_n \right\|_{L_d(0, \infty)}^2
\]

with equality for \( P_n(x) = x^n \).

The following remark is needed concerning the above theorem. Let us choose \( P_n(x) = (x - x_1)^n, x_1 > 0 \). It can be shown easily that the ratio of

\[
\int_0^\infty (P_n'(x))^2 e^{-x} dx / \int_0^\infty (P_n(x))^2 e^{-x} dx
\]

can be made arbitrary small for such polynomials by choosing \( x_1 \) arbitrary large. Thus some condition on the roots such as \( \sum_{k=1}^{n} (x_k)^{-1} > \frac{1}{2} \) is natural.

Next, we turn to the case when \( \omega(x) = (1 - x^2)^{1/2}, [a, b] = [-1, +1] \). Here we prove

**Theorem 2.** Let \( P_n(x) \) be an algebraic polynomial of degree \( n \) having all zeros real and inside \([-1, +1]\). Then for \( n > 2 \) we have

\[
\left\| (1 - x^2)^{1/2} P_n \right\|_{L_d[-1, +1]}^2 > \left( \frac{n}{2} + \frac{1}{4} - \frac{1}{4(n + 1)} \right) \left\| (1 - x^2)^{1/2} P_n \right\|_{L_d[-1, +1]}^2
\]

with equality for \( P_n(x) = (1 - x^2)^m, n = 2m \).

The next two theorems are analogous to Theorem B of P. Erdős [1].
Theorem 3. Let \( P_n(x) \) be an algebraic polynomial of degree \( < n \) having all real roots and no root inside the interval \([-1, +1]\); then we have

\[
\left\| (1 - x^2)^{1/2} P_n'(x) \right\|_{L_2[-1, +1]} \leq \frac{n(n + 1)(2n + 3)}{4(2n + 1)} \left\| (1 - x^2)^{1/2} P_n(x) \right\|_{L_2[-1, +1]}
\]

with equality for \( P_n(x) = (1 + x)^n \) or \( P_n(x) = (1 - x)^n \).

Theorem 4. Let \( x_1, x_2, \ldots, x_n \) be real zeros of an algebraic polynomial \( P_n(x) \) of degree \( < n \). Further, \(-\infty < x_i < 0, i = 1, 2, \ldots, n\). Then

\[
\int_0^\infty xe^{-x}(P_n'(x))^2 \, dx \leq \frac{n}{2(2n + 1)} \int_0^\infty xe^{-x}(P_n(x))^2 \, dx
\]

with equality for \( P_n(x) = x^n \).

2. Proof of Theorem 1. For the proof of this theorem we will need the following facts:

\[
P_n'(x) = P_n(x) \sum_{k=1}^n \frac{1}{x - x_k},
\]

\[
P_n''(x) - P_n(x) P_n''(x) = P_n(x) \sum_{k=1}^n \frac{1}{(x - x_k)^2},
\]

\[
\frac{(x - x_1)^2}{(x - x_k)^2} = -1 + \frac{2(x - x_1)}{x - x_k} + \frac{(x_1 - x_k)^2}{(x - x_k)^2},
\]

\[
P_n'(x) - n \frac{P_n(x)}{x - x_1} = \frac{P_n(x)}{x - x_1} \sum_{k=1}^n \frac{x_k - x_1}{x - x_k}.
\]

From (2.4) and the Cauchy-Schwarz inequality we have

\[
\left( P_n'(x) - n \frac{P_n(x)}{x - x_1} \right)^2 \leq n \frac{P_n'(x)}{(x - x_1)^2} \sum_{k=1}^n \frac{(x_k - x_1)^2}{(x - x_k)^2}.
\]

From (2.5) we immediately have

\[
\int_0^\infty \left( P_n'(x) - n \frac{P_n(x)}{x - x_1} \right)^2 e^{-x} \, dx
\]

\[
\leq n \int_0^\infty \frac{P_n'(x)}{(x - x_1)^2} \sum_{k=1}^n \frac{(x_k - x_1)^2}{(x - x_k)^2} e^{-x} \, dx.
\]

Also, integrating by parts, we have

\[
\int_0^\infty e^{-x} P_n(x) P_n'(x) \, dx = -\frac{1}{2} P_n^2(0) + \frac{1}{2} \int_0^\infty e^{-x} P_n^2(x) \, dx.
\]
Now, we turn to the proof of Theorem 1. On integrating by parts, we have
\[
\int_0^\infty e^{-x}P_n^2(x)dx = -P_n(0)P_n'(0) - \int_0^\infty P_n(x)(P_n''(x) - P_n'(x))e^{-x}dx,
\]
and an equivalent form
\[
2\int_0^\infty e^{-x}P_n^2(x)dx = P_n^2(0) \sum_{k=1}^{n} \frac{1}{x_k} + \int_0^\infty e^{-x}(P_n^2(x) - P_n(x)P_n''(x))dx
+ \int_0^\infty e^{-x}P_n(x)P_n'(x)dx.
\]  
(2.8)

On using (2.2) and (2.7) we obtain
\[
2\int_0^\infty e^{-x}P_n^2(x)dx = P_n^2(0) \left( -\frac{1}{2} + \sum_{k=1}^{n} \frac{1}{x_k} \right) + \frac{1}{2} \int_0^\infty e^{-x}P_n^2(x)dx
+ \int_0^\infty e^{-x}P_n(x)P_n'(x)dx.
\]
(2.9)

Now, let us denote by \(x_1\) the first nonnegative zero of \(P_n(x)\). Then we can write
\[
P_n(x) = (x - x_1)q_{n-1}(x), \quad x_1 > 0,
\]
where \(q_{n-1}(x)\) is a polynomial of degree \(< n - 1\) having real zeros lying inside \([0, \infty)\).

We set
\[
I_1 = \int_0^\infty e^{-x}P_n^2(x) \sum_{k=1}^{n} \frac{1}{x_k} (x - x_k)^2 dx = \int_0^\infty e^{-x}q_{n-1}^2(x) \sum_{k=1}^{n} \frac{(x - x_1)^2}{(x - x_k)^2} dx,
\]
and on using (2.3) we have
\[
I_1 = -n \int_0^\infty e^{-x}q_{n-1}^2(x)dx + 2\int_0^\infty (x - x_1)q_{n-1}^2(x) \sum_{k=1}^{n} \frac{1}{x_k} (x - x_k)^2 e^{-x}dx
+ \int_0^\infty e^{-x}q_{n-1}^2(x) \sum_{k=1}^{n} \frac{(x_1 - x_k)^2}{(x - x_k)^2} dx
= -n \int_0^\infty e^{-x}q_{n-1}^2(x)dx + 2\int_0^\infty P_n'(x)q_{n-1}(x)e^{-x}dx
+ \int_0^\infty e^{-x}q_{n-1}^2(x) \sum_{k=1}^{n} \frac{(x_1 - x_k)^2}{(x - x_k)^2} dx,
\]
\[
I_1 = \frac{1}{n} \int_0^\infty P_n^2(x)e^{-x}dx - \frac{1}{n} \int_0^\infty (P_n'(x) - nq_{n-1}(x))^2 e^{-x}dx
+ \int_0^\infty e^{-x}q_{n-1}^2(x) \sum_{k=1}^{n} \frac{(x_1 - x_k)^2}{(x - x_k)^2} dx;\]
(2.12)
on putting the values of \( I_1 \) from (2.13) into (2.10) we obtain
\[
\left(2 - \frac{1}{n}\right) \int_0^\infty e^{-x} P_n^2(x) \, dx = P_n^2(0) \left(- \frac{1}{2} + \frac{1}{n} \sum_{k=1}^n \frac{1}{x_k}\right) + \frac{1}{2} \int_0^\infty e^{-x} P_n^2(x) \, dx
\]
\[
+ \int_0^\infty e^{-x} q_{n-1}^2(x) \sum_{k=1}^n \frac{(x_k - x_k)^2}{(x - x_k)^2} \, dx
\]
\[
- \frac{1}{n} \int_0^\infty (P_n'(x) - n q_{n-1}(x))^2 e^{-x} \, dx. \quad (2.14)
\]

Now if \( P_n(0) = 0 \) or \( \sum_{k=1}^n (x_k)^{-1} > \frac{1}{2} \) then on using (2.6) we immediately obtain
\[
\left(2 - \frac{1}{n}\right) \int_0^\infty e^{-x} P_n^2(x) \, dx \geq \frac{1}{2} \int_0^\infty e^{-x} P_n^2(x) \, dx,
\]
from which the proof of Theorem 1 is complete. We would like to remark that the ratio of
\[
\int_0^\infty e^{-x} P_n^2(x) \, dx \Big/ \int_0^\infty e^{-x} P_n^2(x) \, dx
\]
does indeed get smaller than \( n/2(2n - 1) \) for polynomials \( P_0(x) = (x - x_0)^n \) for \( x_0 > 2n \). For such polynomials the last two terms in (2.14) are zero and
\[-\frac{1}{2} + \sum_{k=1}^n (x_k)^{-1} < 0.\]

3. Proof of Theorem 2. In order to prove Theorem 2, we need some results proved in [5]. Let \( P_n(x) \) be any polynomial of degree \( n \); then
\[
2 \int_{-1}^1 (1 - x^2) P_n^2(x) \, dx = n \int_{-1}^1 P_n^2(x) \, dx + S_0, \quad (3.1)
\]
\[
(n + 1)(2n + 3) \int_{-1}^1 x^2 P_n^2(x) \, dx = S_2 - S_0 + (n + 1) \int_{-1}^1 P_n^2(x) \, dx
\]
\[
+ 2 \int_{-1}^1 ((1 - x^2) P_n'(x) + nx P_n(x))^2 \, dx \quad (3.2)
\]
where
\[
S_{2i} = \int_{-1}^1 x^{2i} P_n^2(x) \sum_{k=1}^n \frac{(1 - x_k^2)}{(x - x_k)^2} \, dx, \quad i = 0, 1, \ldots . \quad (3.3)
\]
For the proof of (3.1), (3.2) see (2.1) and (2.10) (with \( r = 1 \)) in [5]. Equation (3.2) can be rewritten in the form
\[
(n + 1)(2n + 3) \int_{-1}^1 (1 - x^2) P_n^2(x) \, dx
\]
\[
= S_0 - S_2 + 2(n + 1)^2 \int_{-1}^1 P_n^2(x) \, dx
\]
\[
- 2 \int_{-1}^1 ((1 - x^2) P_n'(x) + nx P_n(x))^2 \, dx.
\]
Therefore

\[(n + 1)(2n + 3) \int_{-1}^{1} (1 - x^2) P_n^2(x) \, dx < S_0 - S_2 + 2(n + 1)^2 \int_{-1}^{1} P_n^2(x) \, dx.\]

(3.4)

From (3.1) and (3.4) we obtain

\[
\frac{2 \int_{-1}^{1} (1 - x^2) P_n^2(x) \, dx}{(n + 1)(2n + 3) \int_{-1}^{1} (1 - x^2) P_n^2(x) \, dx} > \frac{n \int_{-1}^{1} P_n^2(x) \, dx + S_0}{S_0 - S_2 + 2(n + 1)^2 \int_{-1}^{1} P_n^2(x) \, dx}.
\]

(3.5)

It is given that \(|x| < 1, S_0 > 0, S_2 > 0\) and \(S_0 - S_2 > 0\) (here we use (3.3)). Hence it can be verified immediately that

\[
\left( S_0 + n \int_{-1}^{1} P_n^2(x) \, dx \right) / \left( S_0 - S_2 + 2(n + 1)^2 \int_{-1}^{1} P_n^2(x) \, dx \right) > n / 2(n + 1)^2.
\]

(3.6)

From (3.5) and (3.6) we obtain

\[
\int_{-1}^{1} (1 - x^2) P_n^2(x) \, dx > \frac{n(2n + 3)}{4(n + 1)} \int_{-1}^{1} (1 - x^2) P_n^2(x) \, dx.
\]

This proves Theorem 2 as well.

4. Proof of Theorem 3. For the proof of Theorem 3 we need the following lemma.

**Lemma 4.1.** Let \(P_n(x)\) be an algebraic polynomial of degree \(< n\) having real roots and no root in the interval \([-1, +1]\). Then we have

\[
\int_{-1}^{1} (1 - x^2) P_n^2(x) \, dx > \frac{2(2n + 1)}{(n + 1)(2n + 3)} \int_{-1}^{1} P_n^2(x) \, dx.
\]

(4.1)

**Proof** It follows from the well-known theorem [2] that for such \(P_n(x)\) we have

\[
P_n^2(x) = \sum_{p + q = 2n} a_{pq} (1 + x)^p (1 - x)^q, \quad a_{pq} > 0.
\]

(4.2)

Hence we have

\[
\int_{-1}^{1} (1 - x^2) P_n^2(x) \, dx = \sum_{p + q = 2n} a_{pq} \int_{-1}^{1} (1 + x)^p(1 - x)^{q+1} \, dx.
\]
But

\[ \int_{-1}^{1} (1 + x)^{p+1}(1 - x)^{q+1} \, dx \]

\[ = \frac{4(p + 1)(q + 1)}{(p + q + 3)(p + q + 2)} \int_{-1}^{1} (1 + x)^p(1 - x)^q \, dx. \]

Hence

\[ \int_{-1}^{1} (1 - x^2)P_n^2(x) \, dx \]

\[ = 4 \sum_{p+q=2n} q_{pq} \frac{(p + 1)(q + 1)}{(p + q + 3)(p + q + 2)} \int_{-1}^{1} (1 + x)^p(1 - x)^q \, dx \]

\[ > \frac{4(2n + 1)}{(2n + 3)(2n + 2)} \sum_{p+q=2n} q_{pq} \int_{-1}^{1} (1 + x)^p(1 - x)^q \, dx \]

\[ = \frac{2(2n + 1)}{(n + 1)(2n + 3)} \int_{-1}^{1} P_n^2(x) \, dx. \]

This proves the lemma. Now we turn to the proof of the theorem. Since \(|x_k| > 1, k = 1, 2, \ldots, n\), it follows from (3.1) and (3.3) that

\[ 2 \int_{-1}^{1} (1 - x^2)P_n^2(x) \, dx < n \int_{-1}^{1} P_n^2(x) \, dx, \quad (4.3) \]

with equality holding for \(P_n(x) = (1 + x)^p(1 - x)^q, p + q = n\). Now, on using (4.1) and (4.3) we obtain

\[ \frac{\int_{-1}^{1} (1 - x^2)P_n^2(x) \, dx}{\int_{-1}^{1} (1 - x^2)P_n^2(x) \, dx} = \frac{\int_{-1}^{1} P_n^2(x) \, dx}{\int_{-1}^{1} P_n^2(x) \, dx} \frac{\int_{-1}^{1} P_n^2(x) \, dx}{\int_{-1}^{1} (1 - x^2)P_n^2(x) \, dx} \]

\[ < \frac{n}{2} \frac{(n + 1)(2n + 3)}{2(2n + 1)}. \]

This completes the proof of Theorem 3.

5. Proof of Theorem 4. On integrating by parts

\[ \int_{0}^{\infty} P_n^\nu(x)P_n(x)e^{-x} \, dx = - \int_{0}^{\infty} P_n(x)\{P_n^\nu(x) + xP_n(x) - xP_n^\nu(x)\}e^{-x} \, dx \]

\[ = - \int_{0}^{\infty} xe^{-x}(P_n^\nu(x))^2 \, dx + \frac{1}{2}P_n^{\nu(2)}(0) \]

\[ + \frac{1}{2} \int_{0}^{\infty} xe^{-x}P_n^2(x) \, dx - \int_{0}^{\infty} e^{-x}P_n^2(x) \, dx. \quad (5.1) \]
Since \( x_k < 0, k = 1, 2, \ldots, n \), we have
\[
\int_0^\infty xe^{-x}p_n^2(x) \sum_{k=1}^n \frac{1}{(x-x_k)^2} \, dx < \int_0^\infty e^{-x}p_n^2(x) \sum_{k=1}^n \frac{1}{x-x_k} \, dx
\]
\[
= \int_0^\infty e^{-x}p_n(x)P_n'(x) \, dx
\]
\[
= -\frac{1}{2} P_n^2(0) + \frac{1}{2} \int_0^\infty P_n^2(x)e^{-x} \, dx. \tag{5.2}
\]

On using (5.1) and (5.2) and the identity
\[
P_n^2(x) = p_n(x)p_n''(x) + p_n'(x)^2
\]
we obtain
\[
2 \int_0^\infty P_n^2(x)xe^{-x} \, dx < \frac{1}{2} \int_0^\infty xe^{-x}P_n^2(x) \, dx - \frac{1}{2} \int_0^\infty e^{-x}P_n^2(x) \, dx. \tag{5.3}
\]

It is easy to see that under the conditions of our theorem
\[
P_n^2(x) = \sum_{k=0}^{2n} a_k x^k, \quad a_k > 0.
\]
Hence
\[
\int_0^\infty e^{-x}p_n^2(x) \, dx = \sum_{k=0}^{2n} a_k \int_0^\infty e^{-x} x^k \, dx = \sum_{k=0}^{2n} a_k k!,
\]
similarly
\[
\int_0^\infty e^{-x}p_n^2(x) \, dx = \sum_{k=0}^{2n} a_k (k+1)!
\]
\[
< (2n+1) \sum_{k=0}^{2n} a_k k! = (2n+1) \int_0^\infty e^{-x}P_n^2(x) \, dx.
\]

On using (5.3) and the above result it follows that
\[
2 \int_0^\infty (P_n'(x))^2 xe^{-x} \, dx < \left( \frac{1}{2} - \frac{1}{(2n+1)2} \right) \int_0^\infty P_n^2(x)xe^{-x} \, dx.
\]

From this Theorem 4 now follows.

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