SOME RESULTS CONNECTED WITH
A PROBLEM OF ERDŐS. II

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ABSTRACT. It is shown, using the continuum hypothesis, that if E is an uncountable subset of the real line, then there exist subsets \( S_1 \) and \( S_2 \) of the unit interval, such that \( S_1 \) has outer Lebesgue measure one and \( S_2 \) is of the second Baire category and such that neither \( S_1 \) nor \( S_2 \) contains a subset similar (in the sense of elementary geometry) to \( E \). These results are related to a conjecture of P. Erdős.

1. Introduction. P. Erdős [2] presented the following conjecture at the problem session of the Fifth Balkan Mathematical Congress (Belgrade, June 24–30, 1974):

Conjecture. Let \( E \) be an infinite set of real numbers. Then there exists a set of real numbers \( S \) of positive Lebesgue measure which does not contain a set \( \overline{E} \) similar (in the sense of elementary geometry) to \( E \).

If \( E \) is a finite set of real numbers, then every set of real numbers \( S \) of positive Lebesgue measure contains a subset \( E' \) similar to \( E \). This follows from a result of M. S. Ruziewicz [5] or as P. Xenikakis has shown (in a private communication) from Theorem 3 in [3]. By Theorem 4 in [3], the corresponding result holds for Baire sets \( S \) of the second Baire category (\( S \) is a Baire set if it can be written in the form \( S = (G \setminus C) \cup D \), where \( G \) is an open set and \( C \) and \( D \) are sets of the first Baire category).

H. I. Miller and P. Xenikakis [4] have proven the following two theorems related to the conjecture of Erdős.

Theorem A. If \( A \subset \mathbb{R} \) (the real line) possesses the Baire property and is of the second Baire category in \( \mathbb{R} \) and if \( (z_n)_{n=1}^\infty \) is a convergent sequence of reals, then \( A \) contains a set \( A' \) which is similar to the set \( \{z_n; n = 1, 2, \ldots \} \).

Theorem B. If \( A \subset \mathbb{R} \) can be written in the form \( A = (G \setminus C) \cup D \), where \( G \) is a nonempty open set and \( C \) and \( D \) are sets of Lebesgue measure zero and if \( (z_n)_{n=1}^\infty \) is a convergent sequence of reals, then \( A \) contains a set \( A' \) which is similar to the set \( \{z_n; n = 1, 2, \ldots \} \).

The purpose of this work is to show, using transfinite induction and the continuum hypothesis, that if \( E \) is an uncountable subset of the real line, then

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265

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there exist subsets $S_1$ and $S_2$ of the interval $[0, 1]$ such that $m^*(S_1) = 1$ and $S_2$ is of the second Baire category in $R$, and such that neither $S_1$ nor $S_2$ contains a subset similar to $E$. Here $m^*$ denotes outer Lebesgue measure.

2. Results. Our first theorem makes use of the following lemma whose proof can be found in [1].

**Lemma A.** Let $C$ be a closed subset of the real line. Let $B$ be a subset of $C$ such that $B$ has nonempty intersection with every closed subset of $C$ of positive Lebesgue measure. Then $m^*(B) = m(C)$.

We now proceed to prove our first theorem.

**Theorem 1.** If $E$ is an uncountable set of real numbers, then there exists a subset $S$ of $[0, 1]$ such that $m^*(S) = 1$ and no subset of $S$ is similar to $E$.

**Proof.** Let $\mathcal{Q} = \{Q; Q$ a closed subset of $[0, 1]$ and $m(Q) > 0\}$. By the continuum hypothesis $\mathcal{Q}$ can be written in the form $\mathcal{Q} = \{Q_\alpha; \alpha < \Omega\}$ where $\Omega$ denotes the first uncountable ordinal. Similarly $\mathcal{E}$, the family of all subsets of $R$ similar to $E$, can be written in the form $\mathcal{E} = \{E_\alpha; \alpha < \Omega\}$. This is true as there are $c$, the cardinality of the continuum, similarity transformations; since each similarity transformation $f$ is of the form $f(x) = ax + b$. We now proceed to construct two transfinite sequences $\{x_\alpha\}_{\alpha < \omega}$ and $\{y_\alpha\}_{\alpha < \omega}$ of real numbers.

Pick $x_1 \in Q_1$ and $y_1 \in E_1$ such that $x_1 \neq y_1$. Suppose that $\omega$ is an ordinal number $\omega < \Omega$ and that $\{x_\alpha\}_{\alpha < \omega}$ and $\{y_\alpha\}_{\alpha < \omega}$ have been selected such that:

(a) $x_\alpha \in Q_\alpha, y_\alpha \in E_\alpha$ for each $\alpha < \omega$, and

(b) $\{x_\alpha; \alpha < \beta\} \cap \{y_\alpha; \alpha < \beta\} = \emptyset$, for every $\beta, \beta < \omega$.

Then we can find $x_\omega \in Q_\omega$ and $y_\omega \in E_\omega$ such that $\{x_\alpha; \alpha < \omega\} \cap \{y_\alpha; \alpha < \omega\}$ is the empty set, since $Q_\omega$ and $E_\omega$ are uncountable sets and $\omega < \Omega$ implies that $\omega$ is a countable ordinal. Therefore by transfinite sequences $\{x_\alpha\}_{\alpha < \omega}$ and $\{y_\alpha\}_{\alpha < \omega}$ such that $x_\alpha \in Q_\alpha, y_\alpha \in E_\alpha$ for each $\alpha, \alpha < \Omega$, and such that $\{x_\alpha; \alpha < \beta\} \cap \{y_\alpha; \alpha < \beta\} = \emptyset$ for every $\beta, \beta < \Omega$. Let $S$ denote the set $\{x_\alpha; \alpha < \Omega\}$. Then $S \subset [0, 1]$ and by Lemma A we have $m^*(S) = 1$.

Furthermore, no subset of $S$ is similar to $E$. For if some subset, say $S'$, of $S$ is similar to $E$ we have $S' = E'_\gamma$ for some $\gamma < \Omega$. This in turn implies $y_\gamma \in S'$ and hence $y_\gamma \in S$. Therefore there exists $\delta < \Omega$ such that $y_\gamma = x_\delta$ or $\{x_\alpha; \alpha < \beta\} \cap \{y_\alpha; \alpha < \beta\} \neq \emptyset$, where $\beta = \max(\gamma, \delta)$, which is a contradiction.

We need the following lemma in the proof of Theorem 2.

**Lemma B.** Let $\mathcal{Q}$ denote the collection of subsets of $[0, 1]$ given by the formula

\[ \mathcal{Q} = \{(R \setminus \bigcup_{i=1}^{\infty} F_i) \cap [0, 1]|\text{where each } F_i \text{ is a closed and nowhere dense subset of } R\} \]

If $B$ is a subset of $[0, 1]$ and has the property that $Q \cap B \neq \emptyset$ for every $Q \in \mathcal{Q}$, then $B$ is a set of the second Baire category in $R$.

**Proof.** If $B$ is a set of the first Baire category, then we have $B \subset \bigcup_{i=1}^{\infty} \text{Cl}(X_i)$, with each $X_i$ nowhere dense in $R$. This implies that $B \subset \bigcup_{i=1}^{\infty} \text{Cl}(X_i)$, where
Cl denotes the closure operator. From this it follows that \( B \cap (R \setminus \bigcup_{i=1}^{\infty} \text{Cl}(X_i)) = \emptyset \), contradicting the assumption that \( Q \cap B \neq \emptyset \) for every \( Q \in \mathcal{U} \).

**Theorem 2.** If \( E \) is an uncountable set of real numbers, then there exists a subset \( S \) of \([0, 1]\) with the property that no subset of \( S \) is similar to \( E \) and such that \( S \) is of the second Baire category in \( R \).

**Proof.** The proof of Theorem 2 is essentially the same as that of Theorem 1. Here we make use of Lemma B and the fact that the family of sets \( \mathcal{A} \) given in Lemma B can, by assuming the continuum hypothesis, be written in the form \( \mathcal{A} = \{Q_\alpha; \alpha < \Omega\} \), where as before, \( \Omega \) denotes the first uncountable ordinal.

3. **Remark.** Professor John C. Oxtoby has observed that Theorems 1 and 2 can be obtained by applying a theorem of Sierpiński [6] which reads:

**Theorem C.** Given a set \( X \), a family \( \Phi \) of subsets of \( X \), and a group \( G \) of 1-1 mappings of \( X \) onto itself, such that \( \text{card } X = \text{card } \Phi = \text{card } G = \aleph_1 \) and such that \( X \setminus \bigcup_i f_i(H_i) \) is uncountable for each sequence \( \{f_i\} \subset G \) and \( \{H_i\} \subset \Phi \), then there exists an uncountable set \( S \subset X \) with the properties:

- \( H \in \Phi \) implies \( H \cap S \) is countable, and
- \( f \in G \) implies \( f(S) \setminus S \) is countable.

We will sketch Oxtoby’s proof. In the following, let \( \mathcal{E} = \{L; L \) an uncountable subset of \( R \) that contains no uncountable meager subset\} and \( \mathcal{S} = \{S; S \) an uncountable subset of \( R \) that contains no uncountable subset of measure zero\}. Furthermore let \( X = [0, 1], \Psi_1 \) denote all \( G_6 \) nullsets contained in \( X \), \( \Psi_2 \) denote all \( F_\sigma \) meager subsets of \( X \), \( G_1 \) denote all 1-1 Borel measurable and nullset-preserving transformations of \( X \), and \( G_2 \) denote all 1-1 Borel measurable and category-preserving transformations of \( X \). Let \( \mathcal{G} \) denote all sets similar to a given fixed uncountable subset of \( R \). Define

\[
\Phi_1 = \Psi_1 \cup \{E \in \mathcal{E}; E \subset X \text{ and } E \in \mathcal{S}\}
\]

and

\[
\Phi_2 = \Psi_2 \cup \{E \in \mathcal{E}; E \subset X \text{ and } E \in \mathcal{S}\}.
\]

It is easy to verify, by assuming the continuum hypothesis, that the hypotheses of Sierpiński’s theorem are satisfied when we take \( G = G_1 \) and \( \Phi = \Phi_1 \).

The resulting set \( S_1 \subset [0, 1] \) has the following properties:

(i) \( S_1 \in \mathcal{S} \) and hence is nonmeasurable and of the first category on every perfect set;

(ii) \( S_1 \) contains no member of \( \mathcal{G} \);

(iii) \( f \in G_1 \) implies \( f(S_1) \triangle S_1 \) is countable, and

(iv) \( m^*(S_1) = 1 \).

Similarly, taking \( G = G_2 \) and \( \Phi = \Phi_2 \) and by assuming the continuum hypothesis we obtain a set \( S_2 \subset [0, 1] \) with the following properties:
(i) $S_2 \in \mathcal{E}$ and hence is of measure zero and does not possess the Baire property;
(ii) $S_2$ contains no member of $\mathcal{C}$;
(iii) $f \in G_2$ implies $f(S_2) \triangle S_2$ is countable, and
(iv) $S_2$ is of the second category at each point of $[0, 1]$.

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