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SOME RESULTS CONNECTED WITH A PROBLEM OF ERDŐS. II

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ABSTRACT. It is shown, using the continuum hypothesis, that if E is an uncountable subset of the real line, then there exist subsets S_1 and S_2 of the unit interval, such that S_1 has outer Lebesgue measure one and S_2 is of the second Baire category and such that neither S_1 nor S_2 contains a subset similar (in the sense of elementary geometry) to E. These results are related to a conjecture of P. Erdős.

1. Introduction. P. Erdős [2] presented the following conjecture at the problem session of the Fifth Balkan Mathematical Congress (Belgrade, June 24–30, 1974):

Conjecture. Let E be an infinite set of real numbers. Then there exists a set of real numbers S of positive Lebesgue measure which does not contain a set E' similar (in the sense of elementary geometry) to E.

If E is a finite set of real numbers, then every set of real numbers S of positive Lebesgue measure contains a subset E' similar to E. This follows from a result of M. S. Ruziewicz [5] or as P. Xenikakis has shown (in a private communication) from Theorem 3 in [3]. By Theorem 4 in [3], the corresponding result holds for Baire sets S of the second Baire category (S is a Baire set if it can be written in the form $S = (G \setminus C) \cup D$, where G is an open set and C and D are sets of the first Baire category).

H. I. Miller and P. Xenikakis [4] have proven the following two theorems related to the conjecture of Erdős.

THEOREM A. *If $A \subset R$ (the real line) possesses the Baire property and is of the second Baire category in R and if $(z_n)_{n=1}^{\infty}$ is a convergent sequence of reals, then A contains a set A' which is similar to the set $\{z_n; n = 1, 2, \dots\}$.*

THEOREM B. *If $A \subset R$ can be written in the form $A = (G \setminus C) \cup D$, where G is a nonempty open set and C and D are sets of Lebesgue measure zero and if $(z_n)_{n=1}^{\infty}$ is a convergent sequence of reals, then A contains a set A' which is similar to the set $\{z_n; n = 1, 2, \dots\}$.*

The purpose of this work is to show, using transfinite induction and the continuum hypothesis, that if E is an uncountable subset of the real line, then

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there exist subsets S_1 and S_2 of the interval [0, 1] such that $m^*(S_1) = 1$ and S_2 is of the second Baire category in R, and such that neither S_1 nor S_2 contains a subset similar to E. Here m^* denotes outer Lebesgue measure.

2. Results. Our first theorem makes use of the following lemma whose proof can be found in [1].

LEMMA A. *Let C be a closed subset of the real line. Let B be a subset of C such that B has nonempty intersection with every closed subset of C of positive Lebesgue measure. Then $m^*(B) = m(C)$.*

We now proceed to prove our first theorem.

THEOREM 1. *If E is an uncountable set of real numbers, then there exists a subset S of* [0, 1] *such that $m^*(S) = 1$ and no subset of S is similar to E.*

PROOF. Let $\mathfrak{Q} = \{Q;\ Q$ a closed subset of [0, 1] and $m(Q) > 0\}$. By the continuum hypothesis $\mathfrak{Q}$ can be written in the form $\mathfrak{Q} = \{Q_\alpha;\ \alpha < \Omega\}$ where Ω denotes the first uncountable ordinal. Similarly $\mathfrak{S}$, the family of all subsets of R similar to E, can be written in the form $\mathfrak{S} = \{E_\alpha;\ \alpha < \Omega\}$. This is true as there are c, the cardinality of the continuum, similarity transformations; since each similarity transformation f is of the form $f(x) = ax + b$. We now proceed to construct two transfinite sequences $\{x_\alpha\}_{\alpha<\Omega}$ and $\{y_\alpha\}_{\alpha<\Omega}$ of real numbers.

Pick $x_1 \in Q_1$ and $y_1 \in E_1$ such that $x_1 \neq y_1$. Suppose that ω is an ordinal number $\omega < \Omega$ and that $\{x_\alpha\}_{\alpha<\omega}$ and $\{y_\alpha\}_{\alpha<\omega}$ have been selected such that:

(a) $x_\alpha \in Q_\alpha, y_\alpha \in E_\alpha$ for each $\alpha < \omega$, and

(b) $\{x_\alpha;\ \alpha \leqslant \beta\} \cap \{y_\alpha;\ \alpha \leqslant \beta\} = \emptyset$, for every β, $\beta < \omega$.

Then we can find $x_\omega \in Q_\omega$ and $y_\omega \in E_\omega$ such that $\{x_\alpha;\ \alpha \leqslant \omega\} \cap \{y_\alpha;\ \alpha \leqslant \omega\}$ is the empty set, since Q_ω and E_ω are uncountable sets and $\omega < \Omega$ implies that ω is a countable ordinal. Therefore by transfinite sequences $\{x_\alpha\}_{\alpha<\Omega}$ and $\{y_\alpha\}_{\alpha<\Omega}$ such that $x_\alpha \in Q_\alpha$, $y_\alpha \in E_\alpha$ for each α, $\alpha < \Omega$, and such that $\{x_\alpha;\ \alpha \leqslant \beta\} \cap \{y_\alpha;\ \alpha \leqslant \beta\} = \emptyset$ for every β, $\beta < \Omega$. Let S denote the set $\{x_\alpha;\ \alpha < \Omega\}$. Then $S \subset [0, 1]$ and by Lemma A we have $m^*(S) = 1$. Furthermore, no subset of S is similar to E. For if some subset, say S', of S is similar to E we have $S' = E_\gamma$ for some $\gamma < \Omega$. This in turn implies $y_\gamma \in S'$ and hence $y_\gamma \in S$. Therefore there exists $\delta < \Omega$ such that $y_\gamma = x_\delta$ or $\{x_\alpha;\ \alpha \leqslant \beta\} \cap \{y_\alpha;\ \alpha \leqslant \beta\} \neq \emptyset$, where $\beta = \max(\gamma, \delta)$, which is a contradiction.

We need the following lemma in the proof of Theorem 2.

LEMMA B. *Let $\mathfrak{Q}$ denote the collection of subsets of* [0, 1] *given by the formula $\mathfrak{Q} = \{(R \setminus \bigcup_{i=1}^\infty F_i) \cap [0, 1]$ where each F_i is a closed and nowhere dense subset of $R\}$. If B is a subset of* [0, 1] *and has the property that $Q \cap B \neq \emptyset$ for every $Q \in \mathfrak{Q}$, then B is a set of the second Baire category in R.*

PROOF. If B is a set of the first Baire category, then we have $B = \bigcup_{i=1}^\infty X_i$, with each X_i nowhere dense in R. This implies that $B \subset \bigcup_{i=1}^\infty \mathrm{Cl}(X_i)$, where

Cl denotes the closure operator. From this it follows that $B \cap (R \setminus \bigcup_{i=1}^{\infty} \mathrm{Cl}(X_i)) = \varnothing$, contradicting the assumption that $Q \cap B \neq \varnothing$ for every $Q \in \mathfrak{Q}$.

THEOREM 2. *If E is an uncountable set of real numbers, then there exists a subset S of* [0, 1] *with the property that no subset of S is similar to E and such that S is of the second Baire category in R.*

PROOF. The proof of Theorem 2 is essentially the same as that of Theorem 1. Here we make use of Lemma B and the fact that the family of sets $\mathfrak{Q}$ given in Lemma B can, by assuming the continuum hypothesis, be written in the form $\mathfrak{Q} = \{Q_\alpha;\ \alpha < \Omega\}$, where as before, Ω denotes the first uncountable ordinal.

3. Remark. Professor John C. Oxtoby has observed that Theorems 1 and 2 can be obtained by applying a theorem of Sierpiński **[6]** which reads:

THEOREM C. *Given a set X, a family Φ of subsets of X, and a group G of* 1-1 *mappings of X onto itself, such that* card X = card Φ = card $G = \aleph_1$ *and such that $X \setminus \bigcup_i f_i(H_i)$ is uncountable for each sequence $\{f_i\} \subset G$ and $\{H_i\} \subset \Phi$, then there exists an uncountable set $S \subset X$ with the properties*:
$H \in \Phi$ implies $H \cap S$ is countable, and
$f \in G$ implies $f(S) \setminus S$ is countable.

We will sketch Oxtoby's proof. In the following, let $\mathfrak{L} = \{L;\ L$ an uncountable subset of R that contains no uncountable meager subset$\}$ and $\mathfrak{S} = \{S;\ S$ an uncountable subset of R that contains no uncountable subset of measure zero$\}$. Furthermore let $X = [0, 1]$, Ψ_1 denote all G_δ nullsets contained in X, Ψ_2 denote all F_σ meager subsets of X, G_1 denote all 1-1 Borel measurable and nullset-preserving transformations of X, and G_2 denote all 1-1 Borel measurable and category-preserving transformations of X. Let $\mathfrak{E}$ denote all sets similar to a given fixed uncountable subset of R. Define

$$\Phi_1 = \Psi_1 \cup \{E \in \mathfrak{E};\ E \subset X \text{ and } E \in \mathfrak{S}\}$$

and

$$\Phi_2 = \Psi_2 \cup \{E \in \mathfrak{E};\ E \subset X \text{ and } E \in \mathfrak{L}\}.$$

It is easy to verify, by assuming the continuum hypothesis, that the hypotheses of Sierpiński's theorem are satisfied when we take $G = G_1$ and $\Phi = \Phi_1$.

The resulting set $S_1 \subset [0, 1]$ has the following properties:

(i) $S_1 \in \mathfrak{S}$ and hence is nonmeasurable and of the first category on every perfect set;

(ii) S_1 contains no member of $\mathfrak{E}$;

(iii) $f \in G_1$ implies $f(S_1) \triangle S_1$ is countable, and

(iv) $m^*(S_1) = 1$.

Similarly, taking $G = G_2$ and $\Phi = \Phi_2$ and by assuming the continuum hypothesis we obtain a set $S_2 \subset [0, 1]$ with the following properties:

(i) $S_2 \in \mathfrak{L}$ and hence is of measure zero and does not possess the Baire property;

(ii) S_2 contains no member of $\mathfrak{S}$;

(iii) $f \in G_2$ implies $f(S_2) \triangle S_2$ is countable, and

(iv) S_2 is of the second category at each point of [0, 1].

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