A PROBLEM OF GEOMETRY IN $R^n$

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Abstract. Let $\mathcal{F}$ be a finite family of at least $n + 1$ convex sets in the $n$-dimensional Euclidean space $R^n$. Helly's theorem asserts that if all the $(n + 1)$-subfamilies of $\mathcal{F}$ have nonempty intersection, then $\mathcal{F}$ also has nonempty intersection. The main result in this paper is that if almost all of the $(n + 1)$-subfamilies of $\mathcal{F}$ have nonempty intersection, then $\mathcal{F}$ has a subfamily with nonempty intersection containing almost all of the sets in $\mathcal{F}$.

1. Introduction. A family $\mathcal{F}$ of sets is said to be an $I$-family if $\bigcap \mathcal{F} \neq \emptyset$. Let $\mathcal{F}_i$ denote the collection of $I$-subfamilies of $\mathcal{F}$ of size $i$. In this notation, the classic result of Helly [5] may be stated as follows:

Helly's Theorem. If $\mathcal{F}$ is a family of $x$ convex sets in $R^n$ with $x > n$, then $\mathcal{F}$ is an $I$-family if $|\mathcal{F}_{n+1}| = \binom{x}{n+1}$. Counterexamples show that the conclusion is false if $|\mathcal{F}_{n+1}| < \binom{x}{n+1}$.

In this paper, we study the maximal $I$-subfamily of $\mathcal{F}$ when it falls short of the entire family. Greek letters $\alpha$, $\rho$, and $\omega$ always denote real numbers and, unless otherwise stated, we assume that $0 < \alpha, \rho, \omega < 1$.

Let $\mathcal{F}$ be a family of $x$ compact convex sets in $R^n$. The compactness condition is introduced only for convenience as we deal with finite families of sets. We assume that $|\mathcal{F}_r| > \alpha(x)$ for some $\alpha$ and $r$, $n < r < x$. We ask the following questions:

(A) What is the maximal size $\rho x$ of an $I$-subfamily of $\mathcal{F}$?

(B) Does $\rho$ tend to 1 as $\alpha$ tends to 1?

In $R^1$ with $r = 2$, Abbott and Katchalski [1] proved that $\rho = 1 - \sqrt{1 - \alpha}$ and that this result is best possible. The answer to question (B) is thus in the affirmative.

In this paper, we shall deal with the problem in $R^n$ in a more general setting. We consider families of $p$-tuplets in $R^n$, which are defined as unions of $p$ compact convex sets in $R^n$. Our main tools are two lemmas which are given in the next section. In §3, we give a lower bound for $\rho$ in answer to question (A).

The special case $p = 1$ is explored further in §4, leading to an affirmative answer to question (B) for compact convex sets in $R^n$. For $p$-tuplets with $p > 2$, the answer to question (B) is essentially negative. This is pointed out at the end of §3.
For other related problems, consult the comprehensive paper of Danzer, Grünbaum and Klee [2] and the excellent bibliography section in Hadwiger, Debrunner and Klee [4]. For results on p-tuplets, see Grünbaum and Motzkin [3], Katchalski and Liu [6] and Larman [7].

2. Two lemmas. A family \( \mathcal{F} \) of \( x \) sets is said to have property \( (n) \), \( n < x \), if \( \mathcal{F}_n \) is nonempty and there exists a function \( f \) mapping \( \mathcal{F}_n \) into the subsets of \( \bigcup \mathcal{F} \) such that:

1. \( f(\mathcal{A}) \cap (\bigcap \mathcal{B}) \neq \emptyset \) for all \( \mathcal{A} \in \mathcal{F}_n \);
2. if \( \mathcal{B} \) is an \( I \)-subfamily of \( \mathcal{F} \), \( |\mathcal{B}| > n \), then there exists \( \mathcal{A} \in \mathcal{F}_n \) such that \( \mathcal{A} \subset \mathcal{B} \) and \( f(\mathcal{A}) \cap (\bigcap \mathcal{B}) \neq \emptyset \).

**Combinatorial Lemma.** Let \( \mathcal{F} \) be a family of \( x \) sets with property \( (n) \) with \( w = \max \{|f(\mathcal{A})| : \mathcal{A} \in \mathcal{F}_n \} \). If \( |\mathcal{F}_r| > \alpha(\zeta) \) for some \( \alpha \) and \( r, n < r < x \), then \( \mathcal{F} \) has an \( I \)-subfamily of size at least \( t \), where \( t \) is the smallest integer for which \( (r - n)(\zeta) > \alpha(\zeta)/w \).

**Proof.** Define \( g : \mathcal{F}_r \rightarrow \mathcal{F}_n \) in such a way that for \( \mathcal{A} \in \mathcal{F}_r \), \( g(\mathcal{A}) \subset \mathcal{B} \) with \( f \circ g(\mathcal{A}) \cap (\bigcap \mathcal{B}) \neq \emptyset \). Since \( \mathcal{F} \) has property \( (n) \), \( g(\mathcal{A}) \) can always be chosen. If more than one choice is possible, a random selection is made.

Now \( |\mathcal{F}_r| > \alpha(\zeta) \) while \( |\mathcal{F}_n| < (\zeta) \). Hence for some \( \mathcal{A} \in \mathcal{F}_n \), \( f(\mathcal{A}) \cap (\bigcap \mathcal{B}) \neq \emptyset \) for at least \( \alpha(\zeta)/w(\zeta) \) of the \( \mathcal{B} \)'s in \( \mathcal{F}_r \). Now \( |f(\mathcal{A})| < w \). Hence for some \( z \in f(\mathcal{A}) \), \( z \) belongs to \( \bigcap \mathcal{B} \) for at least \( \alpha(\zeta)/w(\zeta) \) of these \( \mathcal{B} \)'s. Let \( z \) belong to \( k \) of the sets in \( \mathcal{F} \). Then it can belong to \( \bigcap \mathcal{B} \) for at most \( (k - n)/(\zeta) \) of the \( \mathcal{B} \)'s in \( \mathcal{F}_r \). Hence we must have \( k > t \) with \( t \) as given in the lemma. The lemma follows immediately. □

Now let \( A \) be any nonempty compact subset of \( R^n \). Define \( h(A) \) to be the point \((a_1, a_2, \ldots, a_n) \in A \) where

\[
a_1 = \{ \max x_1 : (x_1, x_2, \ldots, x_n) \in A \},
\]

and for \( 2 < i < n \),

\[
a_i = \{ \max x_i : (a_1, \ldots, a_{i-1}, x_i, \ldots, x_n) \in A \}.\]

Let \( \Theta \) be the lexicographical order on \( R^n \), that is, \((a_1, a_2, \ldots, a_n) \Theta (b_1, b_2, \ldots, b_n) \) if \( a_i = b_i \) for \( 1 < i < n \) or if there exists some \( k < n \) such that \( a_i = b_i \) for \( i < k \) and \( a_k > b_k \). Note that \( h(A) \Theta h(B) \) if \( A \supset B \), and that \( h((a, b]) = b \) in \( R^1 \).

**Lexicographical Lemma.** Let \( \mathcal{B} \) be a finite \( I \)-family of compact convex sets in \( R^n \) with \( |\mathcal{B}| > n \). Then \( \mathcal{B} \) has a subfamily \( \mathcal{A} \), \( |\mathcal{A}| = n \), such that \( h(\bigcap \mathcal{A}) = h(\bigcap \mathcal{B}) \).

**Proof.** Let \( h(\bigcap \mathcal{B}) = (a_1, a_2, \ldots, a_n) \). Define a subset \( D \) of \( R^n \) by \( D = \{(x_1, x_2, \ldots, x_n) : x_1 > a_1 \} \cup \left( \bigcup_{i=2}^{n-2} \{(a_1, \ldots, a_{i-1}, x_i, \ldots, x_n) : x_i > a_i \} \right) \). It is easy to see that \( D \) is convex and that \( D \cap (\bigcap \mathcal{B}) = \emptyset \).

Let \( \mathcal{B}^* = \mathcal{B} \cup \{D\} \). If every subfamily of \( \mathcal{B}^* \) of size \( n + 1 \) is an \( I \)-subfamily, it will follow from Helly's theorem that \( \mathcal{B}^* \) is an \( I \)-family too,
which it is not. Hence some subfamily $\mathcal{A}^*$ of $\mathcal{B}^*$ of size $n + 1$ has empty intersection. Clearly $D \in \mathcal{A}^*$. Let $\mathcal{A} = \mathcal{A}^* - \{D\}$. Now $|\mathcal{A}| = n$ and $D \cap (\cap \mathcal{A}) = \emptyset$.

It follows from the definition of $D$ that $h(\cap \mathcal{A}) \subseteq h(\cap \mathcal{B})$. On the other hand, $(\cap \mathcal{A}) \supset (\cap \mathcal{B})$ and $h(\cap \mathcal{A}) \subseteq h(\cap \mathcal{B})$. This proves the lemma. \qed

3. \textit{p-tuplets in} $\mathbb{R}^n$. We now state and prove our general result on $p$-tuplets.

\textbf{Theorem A.} For each $\alpha$, there is a $p$ and an $x_0$ such that if $\mathcal{F}$ is a family of $x$ $p$-tuplets in $\mathbb{R}^n$ with $x > x_0$ and $|\mathcal{F}_r| > \alpha(\cdot)$ for some $r$, $n < r < x$, then $\mathcal{F}$ has an $I$-subfamily of size $px$. Furthermore, $p > (\alpha/p^r(\cdot))^{1/(r-n)}$.

\textbf{Proof.} We first show that $\mathcal{F}$ has property $(n)$. $\mathcal{F}_n$ is clearly nonempty. We now verify the two conditions:

1. For $\alpha \in \mathcal{F}_n$, define $f(\alpha) = \{h(A) : A$ a component of $\cap \mathcal{A}\}$. By the definition of $h$, $f(\alpha) \cap (\cap \mathcal{A}) \neq \emptyset$. In fact, $f(\alpha) \subseteq (\cap \mathcal{A})$. We point out that $1/f(\alpha) \leq p^r$ for all $\alpha \in \mathcal{F}_n$.

2. Let $\mathcal{B}$ be any $I$-subfamily of $\mathcal{F}$, $|\mathcal{B}| > n$. Let $B$ be the component of $\cap \mathcal{B}$ which contains $h(\cap \mathcal{B})$. For any $F \in \mathcal{B}$, let $\mathcal{F'}$ be the component of $F$ which contains $\mathcal{B}$ and let $\mathcal{F} = \{\mathcal{F'} : F \in \mathcal{B}\}$. By the lexicographical lemma, there exists $\mathcal{E} \subseteq \mathcal{B}$ such that $|\mathcal{E}| = n$ and $h(\cap \mathcal{E}) = h(\cap \mathcal{B}) = h(B) \in B$. Now $\mathcal{E} = \{F : F \in \mathcal{E} \} \subseteq \mathcal{F}_n$ is contained in $\mathcal{B}$ and $f(\alpha) \cap (\cap \mathcal{B}) \supset \{h(B)\} \neq \emptyset$.

By the combinatorial lemma, $\mathcal{F}$ has an $I$-subfamily of size at least $t$, where $t$ is the smallest integer for which $(r-n)^{\frac{1}{r}} > \alpha(\cdot)/p^r$. A crude estimation yields the desired result when $x > x_0$. \qed

Theorem A is a generalization of an earlier result in $\mathbb{R}^1$ of Katchalski and Liu [6]. In the same paper, it was proved that in $\mathbb{R}^1$ with $p = 2$, $p < (r - 1)/r$ even if we allow $\alpha = 1$. Hence the answer to question (B) for $p$-tuplets is negative for $p > 2$, unless $r$ is sufficiently large, for then the lower bound $(\alpha/p^r(\cdot))^{1/(r-n)}$ is close to 1 if $\alpha$ is.

4. \textit{Compact convex sets in} $\mathbb{R}^n$. In this section, we restrict our attention to compact convex sets in $\mathbb{R}^n$. We state the particular case $p = 1$ of Theorem A as:

\textbf{Theorem B.} For each $\alpha$, there are a $p$ and an $x_0$ such that if $\mathcal{F}$ is a family of $x$ compact convex sets in $\mathbb{R}^n$ with $x > x_0$ and $|\mathcal{F}_r| > \alpha(\cdot)$ for some $r$, $n < r < x$, then $\mathcal{F}$ has an $I$-subfamily of size $px$. Furthermore, $p > (\alpha/\alpha(\cdot))^{1/(r-n)}$.

As it stands, $r$ being fixed, Theorem B does not imply that $p$ tends to 1 as $\alpha$ does. We shall improve the lower bound via a third lemma.

\textbf{Stepping-up Lemma.} Let $\mathcal{F}$ be a family of $x$ convex sets in $\mathbb{R}^n$ such that $|\mathcal{F}_r| > \alpha(\cdot)$ for some $\alpha$ and $r$, $n < r < x$. Then for any $m$, $r < m < x$, $|\mathcal{F}_m| > (1 - (1 - \alpha)^{(x-r)})^{\frac{x}{m}}$. 

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Proof. The number of subfamilies of \( \mathcal{F} \) of size \( r \) which do not belong to \( \mathcal{F}_r \) is at most \((1 - \alpha)\binom{x}{r}\). The number of subfamilies of \( \mathcal{F} \) of size \( m \) containing at least one of these subfamilies of size \( r \) is at most \((1 - \alpha)\binom{\binom{\binom{m}{r}}{r}}{r}\). Since \( m > r > n + 1 \), Helly's theorem shows that the remaining subfamilies of size \( m \) are in \( \mathcal{F}_m \), and there are at least \((1 - (1 - \alpha)\binom{m}{r})\binom{\binom{m}{r}}{r}\) of them. This proves the lemma. \( \square \)

We are now in a position to prove our main result.

Theorem C. For each \( \rho \), there is an \( \alpha \) such that if \( \mathcal{F} \) is a family of \( x \) compact convex sets in \( \mathbb{R}^n \) with \( |\mathcal{F}_r| > \alpha \binom{x}{r} \) for some \( r, n < r < x \), then \( \mathcal{F} \) has an \( I \)-subfamily of size \( \rho x \).

Proof. Choose \( m > r \) such that
\[
\left( \frac{m}{n} \right)^{1/(n-m)} > 1 - \frac{1 - \rho}{2}
\]
and also
\[
\left( \frac{1 + \rho}{2} \right)^{1+1/(m-n)} > \rho.
\]
Once chosen, \( m \) is fixed. Let \( \bar{x} = \max\{ m, x_0 \} \) where \( x_0 \) is as in Theorem B. Let \( \alpha > \max\{ 1 - (1 - \rho)/2(n), 1 - 1/(\bar{x}) \} \). We consider two cases:

(i) \( x < \bar{x} \). We have
\[
|\mathcal{F}_r| > \alpha \binom{x}{r} > \left( 1 - 1/\left( \frac{\bar{x}}{r} \right) \right)\binom{x}{r} = \left( \frac{x}{r} \right) - \left( \binom{x}{r} \right)/\left( \frac{\bar{x}}{r} \right).
\]
It follows that \( |\mathcal{F}_r| = \binom{x}{r} \) and Helly's theorem shows that \( \mathcal{F} \) is an \( I \)-family.

(ii) \( x > \bar{x} \). By the stepping-up lemma,
\[
|\mathcal{F}_m| > \left( 1 - (1 - \alpha)\binom{m}{r} \right)\cdot \binom{x}{m},
\]
and by Theorem B, \( \mathcal{F} \) has an \( I \)-subfamily of size \( \omega x \) where
\[
\omega > \left( \left( 1 - (1 - \alpha)\binom{m}{r} \right)/\binom{m}{n} \right)^{1/(m-n)} = \left( 1 - (1 - \alpha)\binom{m}{r} \right)^{1/(m-n)}\left( \binom{m}{n} \right)^{1/(n-m)} > \left( 1 - \frac{1 - \rho}{2} \right)^{1/(m-n)}\left( 1 - \frac{1 - \rho}{2} \right) = \left( \frac{1 + \rho}{2} \right)^{1+1/(m-n)} > \rho.
\]
This completes the proof of the theorem. \( \square \)

References


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