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A PROBLEM OF GEOMETRY IN R^n

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ABSTRACT. Let $\mathfrak{F}$ be a finite family of at least $n+1$ convex sets in the n-dimensional Euclidean space R^n. Helly's theorem asserts that if all the $(n+1)$-subfamilies of $\mathfrak{F}$ have nonempty intersection, then $\mathfrak{F}$ also has nonempty intersection. The main result in this paper is that if almost all of the $(n+1)$-subfamilies of $\mathfrak{F}$ have nonempty intersection, then $\mathfrak{F}$ has a subfamily with nonempty intersection containing almost all of the sets in $\mathfrak{F}$.

1. Introduction. A family $\mathfrak{F}$ of sets is said to be an I-family if $\bigcap \mathfrak{F} \neq \varnothing$. Let $\mathfrak{F}_i$ denote the collection of I-subfamilies of $\mathfrak{F}$ of size i. In this notation, the classic result of Helly [5] may be stated as follows:

HELLY'S THEOREM. *If $\mathfrak{F}$ is a family of x convex sets in R^n with $x > n$, then $\mathfrak{F}$ is an I-family if $|\mathfrak{F}_{n+1}| = \binom{x}{n+1}$. Counterexamples show that the conclusion is false if $|\mathfrak{F}_{n+1}| < \binom{x}{n+1}$.*

In this paper, we study the maximal I-subfamily of $\mathfrak{F}$ when it falls short of the entire family. Greek letters α, ρ and ω always denote real numbers and, unless otherwise stated, we assume that $0 < \alpha, \rho, \omega < 1$.

Let $\mathfrak{F}$ be a family of x compact convex sets in R^n. The compactness condition is introduced only for convenience as we deal with finite families of sets. We assume that $|\mathfrak{F}_r| \geqslant \alpha\binom{x}{r}$ for some α and r, $n < r < x$. We ask the following questions:

(A) What is the maximal size ρx of an I-subfamily of $\mathfrak{F}$?

(B) Does ρ tend to 1 as α tends to 1?

In R^1 with $r = 2$, Abbott and Katchalski [1] proved that $\rho = 1 - \sqrt{1-\alpha}$ and that this result is best possible. The answer to question (B) is thus in the affirmative.

In this paper, we shall deal with the problem in R^n in a more general setting. We consider families of p-tuplets in R^n, which are defined as unions of p compact convex sets in R^n. Our main tools are two lemmas which are given in the next section. In §3, we give a lower bound for ρ in answer to question (A).

The special case $p = 1$ is explored further in §4, leading to an affirmative answer to question (B) for compact convex sets in R^n. For p-tuplets with $p \geqslant 2$, the answer to question (B) is essentially negative. This is pointed out at the end of §3.

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For other related problems, consult the comprehensive paper of Danzer, Grünbaum and Klee [2] and the excellent bibliography section in Hadwiger, Debrunner and Klee [4]. For results on p-tuplets, see Grünbaum and Motzkin [3], Katchalski and Liu [6] and Larman [7].

2. Two lemmas. A family $\mathfrak{F}$ of x sets is said to have property (n), $n \leqslant x$, if $\mathfrak{F}_n$ is nonempty and there exists a function f mapping $\mathfrak{F}_n$ into the subsets of $\bigcup \mathfrak{F}$ such that:

(1) $f(\mathfrak{A}) \cap (\bigcap \mathfrak{A}) \neq \varnothing$ for all $\mathfrak{A} \in \mathfrak{F}_n$;

(2) if $\mathfrak{B}$ is an I-subfamily of $\mathfrak{F}$, $|\mathfrak{B}| \geqslant n$, then there exists $\mathfrak{A} \in \mathfrak{F}_n$ such that $\mathfrak{A} \subset \mathfrak{B}$ and $f(\mathfrak{A}) \cap (\bigcap \mathfrak{B}) \neq \varnothing$.

COMBINATORIAL LEMMA. *Let $\mathfrak{F}$ be a family of x sets with property (n) with $w = \max\{|f(\mathfrak{A})|: \mathfrak{A} \in \mathfrak{F}_n\}$. If $|\mathfrak{F}_r| \geqslant \alpha\binom{x}{r}$ for some α and r, $n < r < x$, then $\mathfrak{F}$ has an I-subfamily of size at least t, where t is the smallest integer for which $\binom{t-n}{r-n}\binom{x}{n} \geqslant \alpha\binom{x}{r}/w$.*

PROOF. Define g: $\mathfrak{F}_r \to \mathfrak{F}_n$ in such a way that for $\mathfrak{B} \in \mathfrak{F}_r$, $g(\mathfrak{B}) \subset \mathfrak{B}$ with $f \circ g(\mathfrak{B}) \cap (\bigcap \mathfrak{B}) \neq \varnothing$. Since $\mathfrak{F}$ has property (n), $g(\mathfrak{B})$ can always be chosen. If more than one choice is possible, a random selection is made.

Now $|\mathfrak{F}_r| \geqslant \alpha\binom{x}{r}$ while $|\mathfrak{F}_n| \leqslant \binom{x}{n}$. Hence for some $\mathfrak{A} \in \mathfrak{F}_n$, $f(\mathfrak{A}) \cap (\bigcap \mathfrak{B}) \neq \varnothing$ for at least $\alpha\binom{x}{r}/\binom{x}{n}$ of the $\mathfrak{B}$'s in $\mathfrak{F}_r$. Now $|f(\mathfrak{A})| \leqslant w$. Hence for some $z \in f(\mathfrak{A})$, z belongs to $\bigcap \mathfrak{B}$ for at least $\alpha\binom{x}{r}/w\binom{x}{n}$ of these $\mathfrak{B}$'s. Let z belong to k of the sets in $\mathfrak{F}$. Then it can belong to $\bigcap \mathfrak{B}$ for at most $\binom{k-n}{r-n}$ of the $\mathfrak{B}$'s in $\mathfrak{F}_r$. Hence we must have $k \geqslant t$ with t as given in the lemma. The lemma follows immediately. $\square$

Now let A be any nonempty compact subset of R^n. Define $h(A)$ to be the point $(a_1, a_2, \ldots, a_n) \in A$ where

$$a_1 = \{\max x_1\colon (x_1, x_2, \ldots, x_n) \in A\},$$

and for $2 \leqslant i \leqslant n$,

$$a_i = \{\max x_i\colon (a_1, \ldots, a_{i-1}, x_i, \ldots, x_n) \in A\}.$$

Let $\circledcirc$ be the lexicographical order on R^n, that is, $(a_1, a_2, \ldots, a_n) \circledcirc (b_1, b_2, \ldots, b_n)$ if $a_i = b_i$ for $1 \leqslant i \leqslant n$ or if there exists some $k \leqslant n$ such that $a_i = b_i$ for $i < k$ and $a_k > b_k$. Note that $h(A) \circledcirc h(B)$ if $A \supset B$, and that $h([a, b]) = b$ in R^1.

LEXICOGRAPHICAL LEMMA. *Let $\mathfrak{B}$ be a finite I-family of compact convex sets in R^n with $|\mathfrak{B}| \geqslant n$. Then $\mathfrak{B}$ has a subfamily $\mathfrak{A}$, $|\mathfrak{A}| = n$, such that $h(\bigcap \mathfrak{A}) = h(\bigcap \mathfrak{B})$.*

PROOF. Let $h(\bigcap \mathfrak{B}) = (a_1, a_2, \ldots, a_n)$. Define a subset D of R^n by $D = \{(x_1, x_2, \ldots, x_n)\colon x_1 > a_1\} \cup (\bigcup_{i=2}^n \{(a_1, \ldots, a_{i-1}, x_i, \ldots, x_n)\colon x_i > a_i\})$. It is easy to see that D is convex and that $D \cap (\bigcap \mathfrak{B}) = \varnothing$.

Let $\mathfrak{B}^* = \mathfrak{B} \cup \{D\}$. If every subfamily of $\mathfrak{B}^*$ of size $n + 1$ is an I-subfamily, it will follow from Helly's theorem that $\mathfrak{B}^*$ is an I-family too, which it is not. Hence some subfamily $\mathcal{A}^*$ of $\mathcal{B}^*$ of size $n+1$ has empty intersection. Clearly $D \in \mathcal{A}^*$. Let $\mathcal{A} = \mathcal{A}^* - \{D\}$. Now $|\mathcal{A}| = n$ and $D \cap (\bigcap \mathcal{A}) = \varnothing$.

It follows from the definition of D that $h(\bigcap \mathcal{A}) \circledless h(\bigcap \mathcal{B})$. On the other hand, $(\bigcap \mathcal{A}) \supset (\bigcap \mathcal{B})$ and $h(\bigcap \mathcal{A}) \circledgeq h(\bigcap \mathcal{B})$. This proves the lemma. $\square$

3. p-tuplets in R^n. We now state and prove our general result on p-tuplets.

THEOREM A. *For each α, there is a ρ and an x_0 such that if $\mathcal{F}$ is a family of x p-tuplets in R^n with $x \geqslant x_0$ and $|\mathcal{F}_r| \geqslant \alpha\binom{x}{r}$ for some r, $n < r < x$, then $\mathcal{F}$ has an I-subfamily of size ρx. Furthermore, $\rho \geqslant (\alpha/p^n\binom{r}{n})^{1/(r-n)}$.*

PROOF. We first show that $\mathcal{F}$ has property (n). $\mathcal{F}_n$ is clearly nonempty. We now verify the two conditions:

(1) For $\mathcal{A} \in \mathcal{F}_n$, define $f(\mathcal{A}) = \{h(A)$: A a component of $\bigcap \mathcal{A}\}$. By the definition of h, $f(\mathcal{A}) \cap (\bigcap \mathcal{A}) \neq \varnothing$. In fact, $f(\mathcal{A}) \subset (\bigcap \mathcal{A})$. We point out that $|f(\mathcal{A})| \leqslant p^n$ for all $\mathcal{A} \in \mathcal{F}_n$.

(2) Let $\mathcal{B}$ be any I-subfamily of $\mathcal{F}$, $|\mathcal{B}| \geqslant n$. Let B be the component of $\bigcap \mathcal{B}$ which contains $h(\bigcap \mathcal{B})$. For any $F \in \mathcal{B}$, let $\bar{F}$ be the component of F which contains B and let $\overline{\mathcal{B}} = \{\bar{F}$: $F \in \mathcal{B}\}$. By the lexicographical lemma, there exists $\overline{\mathcal{A}} \subset \overline{\mathcal{B}}$ such that $|\overline{\mathcal{A}}| = n$ and $h(\bigcap \overline{\mathcal{A}}) = h(\bigcap \overline{\mathcal{B}}) = h(B) \in B$. Now $\mathcal{A} = \{F$: $\bar{F} \in \overline{\mathcal{A}}\} \in \mathcal{F}_n$ is contained in $\mathcal{B}$ and $f(\mathcal{A}) \cap (\bigcap \mathcal{B}) \supset \{h(B)\} \neq \varnothing$.

By the combinatorial lemma, $\mathcal{F}$ has an I-subfamily of size at least t, where t is the smallest integer for which $\binom{t-n}{r-n}\binom{x}{n} \geqslant \alpha\binom{x}{r}/p^n$. A crude estimation yields the desired result when $x \geqslant x_0$. $\square$

Theorem A is a generalization of an earlier result in R^1 of Katchalski and Liu [6]. In the same paper, it was proved that in R^1 with $p = 2$, $\rho \leqslant (r-1)/r$ even if we allow $\alpha = 1$. Hence the answer to question (B) for p-tuplets is negative for $p \geqslant 2$, unless r is sufficiently large, for then the lower bound $(\alpha/p^n\binom{r}{n})^{1/(r-n)}$ is close to 1 if α is.

4. Compact convex sets in R^n. In this section, we restrict our attention to compact convex sets in R^n. We state the particular case $p = 1$ of Theorem A as:

THEOREM B. *For each α, there are a ρ and an x_0 such that if $\mathcal{F}$ is a family of x compact convex sets in R^n with $x \geqslant x_0$ and $|\mathcal{F}_r| \geqslant \alpha\binom{x}{r}$ for some r, $n < r < x$, then $\mathcal{F}$ has an I-subfamily of size ρx. Furthermore, $\rho \geqslant (\alpha/\binom{r}{n})^{1/(r-n)}$.*

As it stands, r being fixed, Theorem B does not imply that ρ tends to 1 as α does. We shall improve the lower bound via a third lemma.

STEPPING-UP LEMMA. *Let $\mathcal{F}$ be a family of x convex sets in R^n such that $|\mathcal{F}_r| \geqslant \alpha\binom{x}{r}$ for some α and r, $n < r < x$. Then for any m, $r < m < x$, $|\mathcal{F}_m| \geqslant (1-(1-\alpha)\binom{m}{r})\binom{x}{m}$.*

PROOF. The number of subfamilies of $\mathfrak{F}$ of size r which do not belong to $\mathfrak{F}_r$ is at most $(1-\alpha)\binom{x}{r}$. The number of subfamilies of $\mathfrak{F}$ of size m containing at least one of these subfamilies of size r is at most $(1-\alpha)\binom{x}{r}\binom{x-r}{m-r} = (1-\alpha)\binom{x}{m}\binom{m}{r}$. Since $m > r \geqslant n+1$, Helly's theorem shows that the remaining subfamilies of size m are in $\mathfrak{F}_m$, and there are at least $(1-(1-\alpha)\binom{m}{r})\binom{x}{m}$ of them. This proves the lemma. □

We are now in a position to prove our main result.

THEOREM C. *For each ρ, there is an α such that if $\mathfrak{F}$ is a family of x compact convex sets in R^n with $|\mathfrak{F}_r| \geqslant \alpha\binom{x}{r}$ for some r, $n < r < x$, then $\mathfrak{F}$ has an I-subfamily of size ρx.*

PROOF. Choose $m > r$ such that

$$\binom{m}{n}^{1/(n-m)} > 1 - \frac{1-\rho}{2}$$

and also

$$\left(\frac{1+\rho}{2}\right)^{1+1/(m-n)} > \rho.$$

Once chosen, m is fixed. Let $\bar{x} = \max\{m, x_0\}$ where x_0 is as in Theorem B.

Let $\alpha > \max\{1-(1-\rho)/2\binom{m}{r}, 1 - 1/\binom{\bar{x}}{r}\}$. We consider two cases:

(i) $x \leqslant \bar{x}$. We have

$$|\mathfrak{F}_r| \geqslant \alpha\binom{x}{r} > \left(1 - 1/\binom{\bar{x}}{r}\right)\binom{x}{r} = \binom{x}{r} - \binom{x}{r}/\binom{\bar{x}}{r}.$$

It follows that $|\mathfrak{F}_r| = \binom{x}{r}$ and Helly's theorem shows that $\mathfrak{F}$ is an I-family.

(ii) $x > \bar{x}$. By the stepping-up lemma,

$$|\mathfrak{F}_m| \geqslant \left(1 - (1-\alpha)\binom{m}{r}\right) \cdot \binom{x}{m},$$

and by Theorem B, $\mathfrak{F}$ has an I-subfamily of size ωx where

$$\begin{aligned}
\omega &\geqslant \left(\left(1-(1-\alpha)\binom{m}{r}\right)/\binom{m}{n}\right)^{1/(m-n)} \\
&= \left(1-(1-\alpha)\binom{m}{r}\right)^{1/(m-n)}\binom{m}{n}^{1/(n-m)} \\
&> \left(1-\frac{1-\rho}{2}\right)^{1/(m-n)}\left(1-\frac{1-\rho}{2}\right) = \left(\frac{1+\rho}{2}\right)^{1+1/(m-n)} > \rho.
\end{aligned}$$

This completes the proof of the theorem. □

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