THE HOMOLOGY 3-SPHERES WITH INVOLUTIONS

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Abstract. Let $\Sigma$ be a homology 3-sphere which supports an orientation reversing involution. Then $\mu(\Sigma) = 0$.

In this paper we study the homology 3-spheres which support orientation reversing involutions. Given such a homology 3-sphere, we show that it bounds an orientable, parallelizable 4-manifold of index zero. Hence it has Rohlin invariant $0^2$ (Compare this with Problem 4.4 of [3].)

Let $\Sigma$ be a 3-manifold with $H_*(\Sigma) = H_*(S^3)$, and let $g$ be an orientation reversing involution on $\Sigma$ with fixed point set $F$. It follows easily from Smith theory, the Lefschetz fixed point theorem and the slice theorem that $F$ is either a 2-sphere or just two points.

Suppose $F = S^2$. Then by [1, IV. 2.6, p. 179] we know that $\Sigma/Z_2 = \Sigma^*$ is an acyclic 3-manifold with boundary $F$, and it also follows that $\Sigma$ is the boundary of $\Sigma^* \times I$. Hence $\Sigma$ actually bounds an acyclic 4-manifold.

Suppose $F = \{x_0, x_1\}$. Let $D_i$ be a closed $g$-invariant $D^3$-neighborhood of $x_i$ ($i = 0, 1$) and $N = \Sigma - \text{Int}(D_0 \cup D_1)$. Notice that the orbit map $\Pi: N \to N^*$ is a 2-fold covering projection. $\partial N^*$ has two connected components $\partial_+ N^*$ and $\partial_- N^*$. They are both diffeomorphic to $RP^2$. Let $h_+: \partial_+ N^* \to RP^2$ be a diffeomorphism and $h: N^* \to RP^2$ be an extension of $h_+$. (Such an extension always exists.) It can be easily seen that $h|\partial_- N^*$ is homotopic to a diffeomorphism. Define $f: N^* \to RP^2 \times I$ canonically using $h$ and this homotopy. Observe that $f_*: H_*(N^*) \to H_*(RP^2 \times I)$ is an isomorphism. Let $S^H(RP^2 \times I, \partial(RP^2 \times I))$ be the set of all concordance classes of maps $(X, f)$, where $X$ is a 3-manifold with boundary, $f: X \to RP^2 \times I$ is a map which maps $\partial(X)$ diffeomorphically onto $\partial(RP^2 \times I)$ and induces an isomorphism in homology. (Note. $(N^*, f)$ represents an element in $SH(RP^2 \times I, (RP^2 \times I))$.) Following Browder [2] and Wall [6], let

$$\eta: S^H(RP^2 \times I, \partial(RP^2 \times I)) \to [RP^2 \times I, \partial(RP^2 \times I), G/0, *]$$

be the map defined by the normal invariant. Since

$$i^*: [RP^2 \times I, \partial(RP^2 \times I); G/0, *] \to [RP^1 \times I, \partial(RP^1 \times I); G/0, *]$$

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This result was obtained first by J. Birman using different methods. Recently, D. Galewski and R. Stern gave yet another proof.
is an isomorphism, and
\[
[RP^1 \times I, \partial (RP^1 \times I); G/0, \ast] \cong H^2(S(RP^1), \mathbb{Z}_2) = \mathbb{Z}_2.
\]

\([RP^2 \times I, \partial (RP^2 \times I); G/0, \ast]\) has two elements. Of course one of them is represented by the identity map. It follows from [4] or [5, Theorem 2.2] that the nontrivial element in \([RP^2 \times I, \partial (RP^2 \times I); G/0, \ast]\) can be represented by a normal map \((f, b)\), where \(f: RP^2 \times I \to RP^2 \times I\) is a map which induces an isomorphism in homology. Hence \((N^*, f)\) is always normally cobordant rel boundary to a homology equivalence from \(RP^2 \times I\) to itself. Let \((W^*, F)\) be such a normal cobordism. Consider the normal map \(F \times \text{id}: W^4 \times CP^2 \to (RP^2 \times I \times I) \times CP^2\). It gives an 8-dimensional surgery problem with surgery obstruction \(\theta(F \times \text{id}) \in L^8_\ast(\mathbb{Z}_2)\). Let
\[
\bar{F} \times \text{id}: \tilde{W}^4 \times CP^2 \to (S^2 \times I \times I) \times CP^2
\]
be the lifting of \(F \times \text{id}\) in the canonical double covers. \(\bar{F} \times \text{id}\) also gives us a surgery problem with a given homology equivalence on the boundary. Let \(\theta(\bar{F} \times \text{id}) \in L^8_\ast(1)\) be its surgery obstruction, and let \(\rho: L^-_\ast(\mathbb{Z}_2) \to L^8_\ast(1)\) be the transfer homomorphism. Then \(\rho(\theta(F \times \text{id})) = \theta(\bar{F} \times \text{id})\). But \(L^8_\ast(\mathbb{Z}_2) = \mathbb{Z}_2\), \(L^8_\ast(1) = Z\), [6, Theorem 13A.1], so \(\rho: \mathbb{Z}_2 \to Z\) must be the trivial map. Thus \(\theta(\bar{F} \times \text{id}) = 0\). However the 8-dimensional surgery obstruction is given by the index. So \(\text{Index}(\tilde{W} \times CP^2) = 0\). \(\text{Index}(\tilde{W}) = \text{Index}(\tilde{W} \times CP^2)\).

Hence \(\text{Index}(\tilde{W}) = 0\). Clearly \(\tilde{W}\) is parallelizable, and
\[
\partial \tilde{W} = N \cup S^2 \times I \cup S^2 \times I \cup S^2 \times I.
\]

Let \(U = \tilde{W} \cup D^3 \times I \cup D^3 \times I\), where these two copies of \(D^3 \times I\) are attached to \(\tilde{W}\) along the second and third copies of \(S^2 \times I\) above in the obvious fashion. Now \(\partial U = \Sigma \cup S^3\). Fill in \(D^4\) along the \(S^3\) in \(\partial U\), we get a 4-manifold \(M\) with boundary \(\Sigma\). Since \(M\) is constructed from \(\tilde{W}\) by adding two 3-handles and one 4-handle, but still has a nonempty boundary, it is clear that \(M\) is still parallelizable and \(\text{Index}(M) = 0\). In conclusion we have the following:

**Theorem.** Let \(M\) be a 3-manifold with \(H_\ast(\Sigma) = H_\ast(S^3)\). Suppose \(\Sigma\) supports an \(C^\infty\) orientation reversing involution. Then \(\Sigma\) bounds an orientable, parallelizable 4-manifold with index zero. In particular, \(\mu(\Sigma) = 0\).

**References**


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