

WHITNEY'S TRICK FOR THREE 2-DIMENSIONAL HOMOLOGY CLASSES OF 4-MANIFOLDS

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ABSTRACT. In his recent paper, Y. Matsumoto has defined a triple product of 2-homology classes of simply-connected oriented 4-manifolds, when the intersection numbers are zero. In the present paper, the author establishes that three 2-homology classes can be homotopically separated if the intersection numbers and the triple product vanish.

1. Introduction. Let M be a simply-connected oriented 4-manifold possibly with boundary. We shall say that homology classes $x_i \in H_2(M; \mathbb{Z})$, $i = 1, \dots, n$, can be separated, if there exist continuous maps $f_i: S^2 \rightarrow M$ representing x_i , $i = 1, \dots, n$, such that $f_i(S^2) \cap f_j(S^2) = \emptyset$ for $i \neq j$.

In [2], establishing a homotopy version of Whitney's trick in dimension 4, Kobayashi proved that homology classes x_1 and $x_2 \in H_2(M; \mathbb{Z})$ can be separated if and only if the intersection number $x_1 \cdot x_2 = 0$.

On the other hand, Matsumoto [3] defined a "secondary intersection triple", which we shall call Matsumoto triple, $\langle x_1, x_2, x_3 \rangle \in \mathbb{Z}/I$, where x_1, x_2, x_3 are 2-dimensional homology classes of M such that $x_i \cdot x_j = 0$ for $i \neq j$ and I is an ideal of \mathbb{Z} , $\{x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3 \in \mathbb{Z}; y_1, y_2, y_3 \in H_2(M, \mathbb{Z})\}$. See §4 for the definition. He showed that x_1, x_2, x_3 cannot be separated if the triple $\langle x_1, x_2, x_3 \rangle \neq 0$.

In this paper we shall prove the following:

THEOREM. *Let M be a simply-connected oriented 4-manifold, then three homology classes $x_1, x_2, x_3 \in H_2(M; \mathbb{Z})$ can be separated if and only if the intersection numbers $x_i \cdot x_j = 0$ for $i \neq j$ and the Matsumoto triple $\langle x_1, x_2, x_3 \rangle = 0$.*

COROLLARY. *When M is closed, homology classes x_1, x_2, x_3 can be separated if and only if the intersection numbers $x_i \cdot x_j = 0$ for $i \neq j$.*

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2. Some fundamental devices. Throughout this paper, we shall denote by M a simply-connected oriented 4-manifold. Let $f_1, f_2: S^2 \rightarrow M$ be smooth generic immersions in the sense that all the self- and mutual-intersections of

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$S_1 (= f_1(S^2))$ and $S_2 (= f_2(S^2))$ are transversal double points. Suppose that $S_1 \cap S_2$ consists of positive double points, say p_1, \dots, p_m , and negative ones, say q_1, \dots, q_n , where $m, n \neq 0$. Draw smooth arcs γ_1, γ_2 connecting p_1 and q_1 on S_1, S_2 , respectively. We may assume that γ_1 and γ_2 are generic in the sense that they have no self-intersection points and pass through neither the self-intersection points nor the mutual intersection points of S_1 and S_2 . As M is simply-connected, $\gamma_1 \cup \gamma_2$ bounds a smoothly immersed 2-disk Δ , called a Whitney disk, which is generic with respect to S_1 and S_2 . Let ϕ be a nonzero vector field on $\gamma_1 \cup \gamma_2$ such that when restricted to γ_2 it gives a cross-section of a normal 1-vector bundle $\nu(\gamma_2 \hookrightarrow S_2)$ and when restricted to γ_1 the unique extension of $\phi|_{\{p_1, q_1\}}$ over γ_1 which is normal to both S_1 and Δ . Let $\Theta(\Delta) \in Z = \pi_1(SO_2)$ be the obstruction to extending ϕ over Δ . We shall say that the Whitney disk Δ is good, if $\Theta(\Delta) = 0$.

Device 1 (Making the Whitney disk Δ good). When the Whitney disk Δ is not good, one can obtain a good Whitney disk spanning $\gamma_1 \cup \gamma_2$ by spinning Δ around γ_1 or γ_2 (see [1]).

Device 2 (Making S_1 escape from the intersection with $\text{int } \Delta$ across γ_1). If $S_1 \cap \text{int } \Delta \neq \emptyset$, for a point $p \in S_1 \cap \text{int } \Delta$ we take a point $p' \in \text{int } \gamma_1$ and a simple arc γ connecting p and p' on Δ such that $\gamma \cap S_1 = \{p, p'\}$ and $\gamma \cap S_2 = \emptyset$. Pushing a neighborhood of the intersecting point p in S_1 along the arc γ off Δ as in Figure 1, we can make S_1 escape from the intersection point p with $\text{int } \Delta$ across γ_1 , by adding two self-intersection points with opposite sign for S_1 .

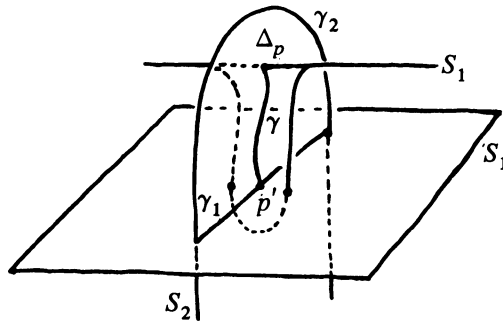


FIGURE 1

Device 3 (Whitney's trick for S_1 and S_2 across γ_1). Let Δ be a good immersed Whitney disk such that $S_1 \cap \text{int } \Delta = \emptyset$. We set

$$D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1, y > 0\},$$

$$D' = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < (6/5)^2, y > 0\},$$

$$C_1 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1, y > 0\},$$

$$C'_1 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = (6/5)^2, y > 0\},$$

$$C_2 = \{(x, 0) \in R^2; -1 < x < 1\},$$

$$C'_2 = \{(x, 0) \in R^2; -6/5 < x < 6/5\}.$$

Let $f: D \rightarrow M$ be an immersion such that $f(D) = \Delta, f(C_1) = \gamma_1, f(C_2) = \gamma_2$. Adding a collar along γ_1 , we have an extension $f': D' \rightarrow M$ of f , so that $f'(C'_1) \cap S_1 = \emptyset, f'(C'_2) \subset S_2$. Since Δ is good, we have an immersion $\tilde{f}: D' \times [-1, 1] \rightarrow M$ using the vector field $\tilde{\phi}$ (the extension of ϕ over Δ) such that \tilde{f} restricted to $D' = D' \times 0$ coincides with f' . Now the immersed 2-sphere S_2 shall be modified as follows.

$$S'_2 = (S_2 - \tilde{f}(C'_2 \times [-1, 1])) \cup \tilde{f}(C'_1 \times [-1, 1]) \cup \tilde{f}(D' \times \{-1, 1\}).$$

Rounding the corners, one can assume S'_2 is a generic immersed 2-sphere. Now $S_1 \cap S'_2 = \{p_2, \dots, p_m, q_2, \dots, q_n\}$. It is easy to construct a generic immersion $f'_2: S^2 \rightarrow M$ with $f'_2(S^2) = S'_2$ which is regularly homotopic to f_2 . This process will be referred to as Whitney's trick for S_1 and S_2 across γ_1 (see [4, Theorem 6.6]).

REMARK. (1) Let X be a compact subset of M . If $\Delta \cap X = \emptyset$, then $S'_2 \cap X = S_2 \cap X$ in Device 3.

(2) These can be applied to generic intersections of immersed disks and spheres not only of immersed spheres.

Using the Devices 1, 2 and 3 repeatedly, we obtain

PROPOSITION (KOBAYASHI [2]). Let $x_1, x_2 \in H_2(M; Z)$ be homology classes such that $x_1 \cdot x_2 = r$. Then x_1 and x_2 can be represented by continuous maps of S^2 whose images have $|r|$ points in common. In particular, if $x_1 \cdot x_2 = 0$, x_1 and x_2 can be separated.

3. The key lemma. Let S_1, S_2, S_3 be smoothly immersed generic 2-spheres in M such that their mutual algebraic intersection numbers are all zero. We denote by $p_\lambda^{(i,j)}$ (or $q_\lambda^{(i,j)}$) the λ th positive (or negative) intersection point of S_i and S_j . Draw a smoothly imbedded arc $\gamma_{\lambda,i}^{(i,j)}$ (or $\gamma_{\lambda,j}^{(i,j)}$) connecting $p_\lambda^{(i,j)}$ and $q_\lambda^{(i,j)}$ on the immersed sphere S_i (or S_j). We assume that $\gamma_{\lambda,i}^{(i,j)} \cap \gamma_{\mu,i}^{(i,k)} = \emptyset$ ($\lambda \neq \mu$ or $j \neq k$). Let $\Delta_\lambda^{(i,j)}$ be a smoothly immersed generic 2-disk bounding the circle $\gamma_{\lambda,i}^{(i,j)} \cup \gamma_{\lambda,j}^{(i,j)}$.

LEMMA. Suppose that $\Delta_1^{(i,j)} \cap S_k = \{a_1, \dots, a_m\}$ and $\Delta_2^{(i,j)} \cap S_k = \{b_1, \dots, b_n\}$ where $\{i, j, k\} = \{1, 2, 3\}$. Then one can regularly homotope S_1, S_2, S_3 to obtain S'_1, S'_2, S'_3 and Whitney disks $\{\Delta'_\lambda^{(i,j)}\}$ such that:

- (1) $S'_i \cap S'_j = S_i \cap S_j (\forall i, j)$,
- (2) $\Delta'_\lambda^{(i,j)} \cap S'_k = \Delta_\lambda^{(i,j)} \cap S_k$, for $\lambda > 2$, and
- (3) $\Delta'_1^{(i,j)} \cap S'_k = \{a_2, \dots, a_m\}, \Delta'_2^{(i,j)} \cap S'_k = \{b_0, b_1, \dots, b_n\}$ where b_0 and a_1 have the same sign.

PROOF. (See Figure 2.) Make S_k escape from the intersection point a_1 with $\Delta_1^{(i,j)}$ across $\gamma_{1,i}^{(i,j)}$, adding new intersections of S_k and $S_i, p_0^{(k,i)}$ and $q_0^{(k,i)}$; then we obtain a small Whitney disk Δ' . Choose an imbedding $g: B = [-1, 1] \times [0, 1] \rightarrow S_i$ such that

$$g([-1, 1] \times \{0\}) = \partial\Delta' \cap S_i, \quad g(\{1\} \times \{0\}) = p_0^{(k,i)},$$

$$g(\{-1\} \times \{0\}) = q_0^{(k,i)}, \quad g(B) \cap \gamma_{\lambda,i}^{(i,j)} = \emptyset$$

except for $l = j$ and $\lambda = 1$ or 2 , and

$$g(B) \cap \gamma_{1,i}^{(i,j)} = \gamma_{1,i}^{(i,j)}\left(\left[0, \frac{1}{2}\right]\right) = g_1, \quad g(B) \cap \gamma_{2,i}^{(i,j)} = \gamma_{2,i}^{(i,j)}\left(\left[\frac{1}{2}, 1\right]\right) = g_2,$$

where

$$\gamma_{1,i}^{(i,j)}(0) = p_1^{(i,j)}, \quad \gamma_{2,i}^{(i,j)}(1) = q_2^{(i,j)}.$$

Let $\gamma_{0,k}^{(k,i)} = \partial\Delta' \cap S_k$,

$$\gamma_{0,i}^{(k,i)} = g(\{1, -1\} \times [0, 1] \cup [-1, 1] \times \{1\}),$$

and $\tilde{\Delta}_0^{(k,i)} = g(B) \cup \Delta'$. Then $\tilde{\Delta}_0^{(k,i)} \cap S_j = \{p_1^{(i,j)}, q_2^{(i,j)}\}$. Let ψ be a vector field on the arc $g(\{0\} \times [0, 1])$ which does not lie in $T(S_i)$, $T(S_i)|_{g_1} + T(\Delta_l^{(i,j)})|_{g_l}$ ($l = 1, 2$). Push $\tilde{\Delta}_0^{(k,i)}$ off S_j along ψ keeping $\gamma_{0,i}^{(k,i)}$ fixed; then we obtain an imbedded disk $\tilde{\Delta}_0^{(k,i)}$ bounded by $\gamma_0^{(k,i)} \cup \gamma_{0,k}^{(k,i)}$ such that it meets S_i normally along $\gamma_{0,i}^{(k,i)}$, S_k normally along $\gamma_{0,k}^{(k,i)}$, and

$$S_i \cap \text{int } \tilde{\Delta}_0^{(k,i)} = \emptyset, \quad S_k \cap \text{int } \tilde{\Delta}_0^{(k,i)} = \emptyset,$$

$$S_j \cap \tilde{\Delta}_0^{(k,i)} = \{p, q\}, \quad \tilde{\Delta}_0^{(k,i)} \cap \Delta_\lambda^{(i,j)} = \emptyset$$

for $\lambda = 1, 2$. We can cancel these intersection points p and q as follows. Let γ (or γ') be a generic arc connecting p and q on $\tilde{\Delta}_0^{(k,i)}$ (or S_j), and let Δ be a good generic immersed disk bounded by $\gamma \cup \gamma'$. We can make S_j escape from the intersection with $\text{int } \Delta$ across γ' . Doing Whitney's trick for $\tilde{\Delta}_0^{(k,i)}$ and S_j across γ' , we obtain a new immersed disk $\Delta_0^{(k,i)}$ such that $S_j \cap \Delta_0^{(k,i)} = \emptyset$. We may assume that $\Delta_0^{(k,i)}$ is good, and

$$\Delta_0^{(k,i)} \cap \Delta_\lambda^{(j,k)} = \emptyset, \quad \text{int } \Delta_0^{(k,i)} \cap \text{int } \Delta_\mu^{(i,j)} = \emptyset.$$

For example, if $\Delta_0^{(k,i)} \cap \Delta_\lambda^{(j,k)} \neq \emptyset$, we can make $\Delta_0^{(k,i)}$ escape from the intersection with $\Delta_\lambda^{(j,k)}$ across $\gamma_{\lambda,k}^{(j,k)}$ by adding two intersection points of $\Delta_0^{(k,i)}$ and S_k . Possibly $\text{int } \Delta_0^{(k,i)} \cap S_k \neq \emptyset$, $\text{int } \Delta_0^{(k,i)} \cap S_i \neq \emptyset$. Make $\Delta_0^{(k,i)}$ escape from this intersection with S_k across $\gamma_{0,k}^{(k,i)}$, if necessary.

Using $\Delta_0^{(k,i)}$, we can do Whitney's trick for S_k and S_i across $\gamma_{0,k}^{(k,i)}$ and we obtain a new immersed 2-sphere S'_i such that

$$S'_i \cap S_k = S_i \cap S_k - \{p_0^{(k,i)}, q_0^{(k,i)}\} \quad \text{and} \quad \Delta_\lambda^{(j,k)} \cap S'_i = \Delta_\lambda^{(j,k)} \cap S_i.$$

Let \tilde{f} denote the immersion: $D' \times [-1, 1] \rightarrow M$ in Device 3 such that $\tilde{f}(C_2 \times \{0\}) = \gamma_{0,i}^{(k,i)}$. We may assume that

$$\tilde{f}(\{(0, 0)\} \times [-1, 1]) = \gamma_{2,i}^{(i,j)} \cap \tilde{f}(D' \times [-1, 1]).$$

We shall modify the disk $\Delta_2^{(i,j)}$ as follows:

$$\Delta_2^{(i,j)} = \Delta_2^{(i,j)} \cup \tilde{f}(\{(x, 0); 0 < x < 6/5\} \times [-1, 1]).$$

Then we obtain a new intersection point b_0 . Q.E.D.

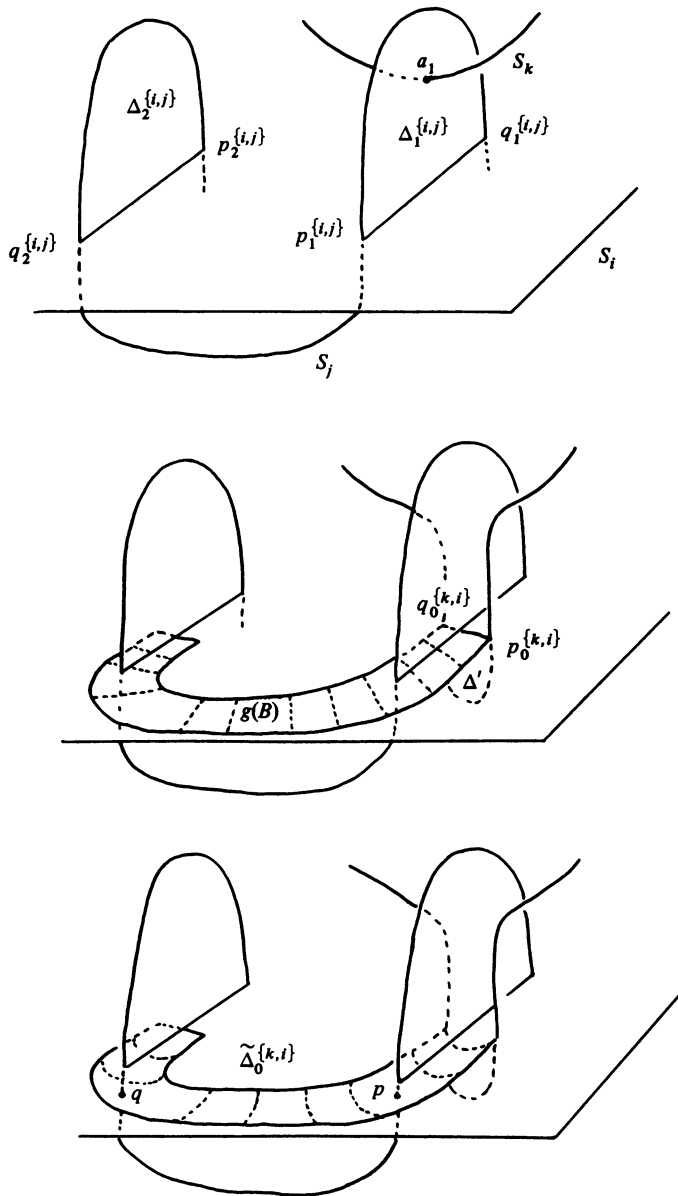


FIGURE 2

4. Proof of theorem and corollary. Let $x_1, x_2, x_3 \in H_2(M; \mathbb{Z})$ be homology classes such that $x_i \cdot x_j = 0$ for $i \neq j$. Represent x_1, x_2, x_3 by smoothly immersed generic 2-spheres S_1, S_2, S_3 , and let $p_\lambda^{(i,j)}, \gamma_{\lambda,i}^{(i,j)}, \Delta_\lambda^{(i,j)}$ be as in §3, but we do not require the condition $\gamma_{\lambda,i}^{(i,j)} \cap \gamma_{\mu,i}^{(i,k)} = \emptyset$ ($\lambda \neq \mu$ or $j \neq k$). The Whitney disk $\Delta_\lambda^{(i,j)}$ is oriented as in Figure 3. Now the Matsumoto triple $\langle x_1, x_2, x_3 \rangle$ is defined as follows:

$$\begin{aligned} \langle x_1, x_2, x_3 \rangle &= \sum_{\lambda} S_1 \cdot \Delta_{\lambda}^{\{2,3\}} + \sum_{\mu} S_2 \cdot \Delta_{\mu}^{\{3,1\}} + \sum_{\nu} S_3 \cdot \Delta_{\nu}^{\{1,2\}} \\ &+ \sum_{\mu, \nu} \frac{\partial \Delta_{\mu}^{\{3,1\}} \cdot \partial \Delta_{\nu}^{\{1,2\}}}{S_1} + \sum_{\nu, \lambda} \frac{\partial \Delta_{\nu}^{\{1,2\}} \cdot \partial \Delta_{\lambda}^{\{2,3\}}}{S_2} \\ &+ \sum_{\lambda, \mu} \frac{\partial \Delta_{\lambda}^{\{2,3\}} \cdot \partial \Delta_{\mu}^{\{3,1\}}}{S_3} \pmod I, \end{aligned}$$

where $S_1 \cdot \Delta_{\lambda}^{\{2,3\}}$, etc., denote the intersection number of S_1 and $\Delta_{\lambda}^{\{2,3\}}$, etc., and $(\partial \Delta_{\mu}^{\{3,1\}} \cdot \partial \Delta_{\nu}^{\{1,2\}})/S_1$, etc. denote the intersection number of $\partial \Delta_{\mu}^{\{3,1\}}$ and $\partial \Delta_{\nu}^{\{1,2\}}$ on S_1 , etc.

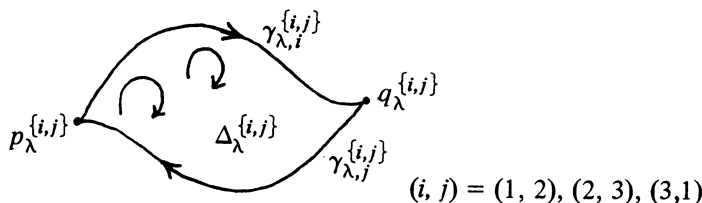


FIGURE 3

PROOF OF THEOREM. Let $x_1, x_2, x_3 \in H_2(M; Z)$ be homology classes such that $x_i \cdot x_j = 0$ for $i \neq j$ and $\langle x_1, x_2, x_3 \rangle = 0 \in Z/I$. Let $S_i, \gamma_{\lambda,i}^{\{i,j\}}, \Delta_{\lambda}^{\{i,j\}}$ be as above. Now we assume as in §3 that $\gamma_{\lambda,i}^{\{i,j\}} \cap \gamma_{\mu,i}^{\{i,k\}} = \emptyset$ ($\lambda \neq \mu$ or $j \neq k$); then the Matsumoto triple $\langle x_1, x_2, x_3 \rangle$ is defined as

$$\sum_{\lambda} S_1 \cdot \Delta_{\lambda}^{\{2,3\}} + \sum_{\mu} S_2 \cdot \Delta_{\mu}^{\{3,1\}} + \sum_{\nu} S_3 \cdot \Delta_{\nu}^{\{1,2\}}.$$

We may assume that this sum is zero. In fact, if the ideal I is $\{0\}$, it is always zero. If I is not $\{0\}$, there exist homology classes $y_1, y_2, y_3 \in H_2(M; Z)$ such that $\langle x_1, x_2, x_3 \rangle = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$. Let F_1, F_2, F_3 be immersed 2-spheres representing y_1, y_2, y_3 . Make connected-sums of $\Delta_{\lambda}^{\{2,3\}}$ and $-F_1, \Delta_{\lambda}^{\{3,1\}}$ and $-F_2, \Delta_{\lambda}^{\{1,2\}}$ and $-F_3$, where $-F$ is an immersed 2-sphere with the reversed orientation. Then if we use the resulting immersed disks instead of $\Delta_{\lambda}^{\{2,3\}}, \Delta_{\lambda}^{\{3,1\}}, \Delta_{\lambda}^{\{1,2\}}$, the sum is zero. We may assume that every Whitney disk is good in the sense of §2 and that there is no mutual-intersection of Whitney disks (and even there is no self-intersection of Whitney disks, i.e. every Whitney disk is an imbedded disk). (See proof of lemma.) As $x_1 \cdot x_3 = 0$, we may assume $S_1 \cap S_3 = \emptyset$ by the proposition. Escaping the intersection $S_1 \cap \text{int } \Delta_{\lambda}^{\{1,2\}}$ across $\gamma_{\lambda,i}^{\{1,2\}}$, we obtain a new immersed 2-sphere S'_1 , so that $S'_1 \cap \text{int } \Delta_{\lambda}^{\{1,2\}} = \emptyset$ and $S'_1 \cap S_3 = \emptyset$. Using $\Delta_{\lambda}^{\{1,2\}}$, do Whitney's trick for S'_1 and S_2 across $\gamma_{\lambda,i}^{\{1,2\}}$, and we shall obtain a new immersed 2-sphere S'_2 such that $S'_1 \cap S'_2 = \emptyset$. By Lemma, we may assume that $S'_1 \cap \Delta_{\lambda}^{\{2,3\}} = \emptyset$ for $\lambda \neq 1$. Then $S'_1 \cdot \Delta_{\lambda}^{\{2,3\}} = 0$. Using Devices 1, 2 and 3, we obtain $S'_1 \cap \Delta^{\{2,3\}} = \emptyset$. Now we can do Whitney's trick for S'_2 and S_3 (using Device 2), and we obtain the required maps. Q.E.D.

PROOF OF COROLLARY. If one of x_1, x_2, x_3 is 0, then this follows immediately from the proposition. If one of x_1, x_2, x_3 , say x_1 , is a primitive element, i.e. there is no homology class $x \in H_2(M; Z)$ such that $x_1 = mx$ ($m \in Z, \neq 1, -1$), then there exists a homology class $y \in H_2(M; Z)$ such that $x_1 \cdot y = 1$. Therefore $\langle x_1, x_2, x_3 \rangle = 0 \pmod{I = (1)}$ and they can be separated by the theorem. If x_1, x_2, x_3 can be separated, also mx_1, x_2, x_3 can be separated by using the "self-connected-sum" of the immersed 2-sphere representing x_i ($m \in Z$). Q.E.D.

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