WHITNEY'S TRICK FOR THREE 2-DIMENSIONAL HOMOLOGY CLASSES OF 4-MANIFOLDS

MASAYUKI YAMASAKI

ABSTRACT. In his recent paper, Y. Matsumoto has defined a triple product of 2-homology classes of simply-connected oriented 4-manifolds, when the intersection numbers are zero. In the present paper, the author establishes that three 2-homology classes can be homotopically separated if the intersection numbers and the triple product vanish.

1. Introduction. Let M be a simply-connected oriented 4-manifold possibly with boundary. We shall say that homology classes $x_i \in H_2(M; Z)$, $i = 1, \ldots, n$, *can be separated*, if there exist continuous maps $f_i: S^2 \to M$ representing x_i, $i = 1, \ldots, n$, such that $f_i(S^2) \cap f_j(S^2) = \varnothing$ for $i \neq j$.

In [2], establishing a homotopy version of Whitney's trick in dimension 4, Kobayashi proved that homology classes x_1 and $x_2 \in H_2(M; Z)$ can be separated if and only if the intersection number $x_1 \cdot x_2 = 0$.

On the other hand, Matsumoto [3] defined a "secondary intersection triple", which we shall call Matsumoto triple, $\langle x_1, x_2, x_3 \rangle \in Z/I$, where x_1, x_2, x_3 are 2-dimensional homology classes of M such that $x_i \cdot x_j = 0$ for $i \neq j$ and I is an ideal of Z, $\{x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3 \in Z; y_1, y_2, y_3 \in H_2(M, Z)\}$. See §4 for the definition. He showed that x_1, x_2, x_3 cannot be separated if the triple $\langle x_1, x_2, x_3 \rangle \neq 0$.

In this paper we shall prove the following:

THEOREM. *Let M be a simply-connected oriented 4-manifold, then three homology classes $x_1, x_2, x_3 \in H_2(M; Z)$ can be separated if and only if the intersection numbers $x_i \cdot x_j = 0$ for $i \neq j$ and the Matsumoto triple $\langle x_1, x_2, x_3 \rangle = 0$.*

COROLLARY. *When M is closed, homology classes x_1, x_2, x_3 can be separated if and only if the intersection numbers $x_i \cdot x_j = 0$ for $i \neq j$.*

The author is grateful to Professor M. Kato for helpful suggestions and advice without which this paper could not have been completed.

2. Some fundamental devices. Throughout this paper, we shall denote by M a simply-connected oriented 4-manifold. Let $f_1, f_2: S^2 \to M$ be smooth generic immersions in the sense that all the self- and mutual-intersections of

---

$S_1 (= f_1(S^2))$ and S_2 $(= f_2(S^2))$ are transversal double points. Suppose that $S_1 \cap S_2$ consists of positive double points, say $p_1, \ldots, p_m$, and negative ones, say $q_1, \ldots, q_n$, where $m, n \neq 0$. Draw smooth arcs γ_1, γ_2 connecting p_1 and q_1 on S_1, S_2, respectively. We may assume that γ_1 and γ_2 are generic in the sense that they have no self-intersection points and pass through neither the self-intersection points nor the mutual intersection points of S_1 and S_2. As M is simply-connected, $\gamma_1 \cup \gamma_2$ bounds a smoothly immersed 2-disk Δ, called a Whitney disk, which is generic with respect to S_1 and S_2. Let ϕ be a nonzero vector field on $\gamma_1 \cup \gamma_2$ such that when restricted to γ_2 it gives a cross-section of a normal 1-vector bundle $\nu(\gamma_2 \hookrightarrow S_2)$ and when restricted to γ_1 the unique extension of $\phi|_{\{p_1, q_1\}}$ over γ_1 which is normal to both S_1 and Δ. Let $\theta(\Delta) \in Z = \pi_1(SO_2)$ be the obstruction to extending ϕ over Δ. We shall say that the Whitney disk Δ is good, if $\theta(\Delta) = 0$.

  Device 1 (*Making the Whitney disk* Δ *good*). When the Whitney disk Δ is not good, one can obtain a good Whitney disk spanning $\gamma_1 \cup \gamma_2$ by spinning Δ around γ_1 or γ_2 (see [1]).

  Device 2 (*Making* S_1 *escape from the intersection with* int Δ *across* γ_1). If $S_1 \cap$ int $\Delta \neq \varnothing$, for a point $p \in S_1 \cap$ int Δ we take a point $p' \in$ int γ_1 and a simple arc γ connecting p and p' on Δ such that $\gamma \cap S_1 = \{p, p'\}$ and $\gamma \cap S_2 = \varnothing$. Pushing a neighborhood of the intersecting point p in S_1 along the arc γ off Δ as in Figure 1, we can make S_1 escape from the intersection point p with int Δ across γ_1, by adding two self-intersection points with opposite sign for S_1.
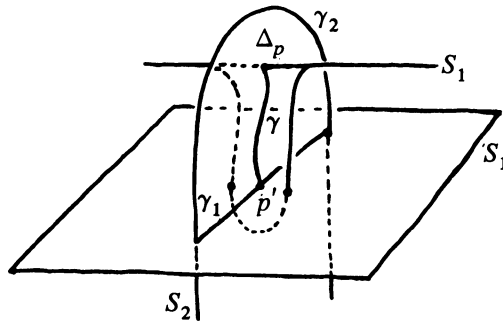

FIGURE 1

  Device 3 (*Whitney's trick for* S_1 *and* S_2 *across* γ_1). Let Δ be a good immersed Whitney disk such that $S_1 \cap$ int $\Delta = \varnothing$. We set

$$D = \{(x, y) \in R^2; x^2 + y^2 < 1, y > 0\},$$
$$D' = \{(x, y) \in R^2; x^2 + y^2 < (6/5)^2, y > 0\},$$
$$C_1 = \{(x, y) \in R^2; x^2 + y^2 = 1, y > 0\},$$
$$C_1' = \{(x, y) \in R^2; x^2 + y^2 = (6/5)^2, y > 0\},$$

$$C_2 = \{(x, 0) \in R^2; \ -1 < x < 1\},$$

$$C_2' = \{(x, 0) \in R^2; \ -6/5 < x < 6/5\}.$$

Let $f: D \to M$ be an immersion such that $f(D) = \Delta$, $f(C_1) = \gamma_1$, $f(C_2) = \gamma_2$. Adding a collar along γ_1, we have an extension $f': D' \to M$ of f, so that $f'(C_1') \cap S_1 = \varnothing$, $f'(C_2') \subset S_2$. Since Δ is good, we have an immersion $\tilde{f}$: $D' \times [-1, 1] \to M$ using the vector field $\tilde{\phi}$ (the extension of ϕ over Δ) such that $\tilde{f}$ restricted to $D' = D' \times 0$ coincides with f'. Now the immersed 2-sphere S_2 shall be modified as follows.

$$S_2' = \left(S_2 - \tilde{f}(C_2' \times [-1, 1]) \right) \cup \tilde{f}(C_1' \times [-1, 1]) \cup \tilde{f}(D' \times \{-1, 1\}).$$

Rounding the corners, one can assume S_2' is a generic immersed 2-sphere. Now $S_1 \cap S_2' = \{p_2, \ldots, p_m, q_2, \ldots, q_n\}$. It is easy to construct a generic immersion $f_2': S^2 \to M$ with $f_2'(S^2) = S_2'$ which is regularly homotopic to f_2. This process will be referred to as Whitney's trick for S_1 and S_2 across γ_1 (see [4, Theorem 6.6]).

REMARK. (1) Let X be a compact subset of M. If $\Delta \cap X = \varnothing$, then $S_2' \cap X = S_2 \cap X$ in Device 3.

(2) These can be applied to generic intersections of immersed disks and spheres not only of immersed spheres.

Using the Devices 1, 2 and 3 repeatedly, we obtain

PROPOSITION (KOBAYASHI [2]). *Let $x_1, x_2 \in H_2(M; Z)$ be homology classes such that $x_1 \cdot x_2 = r$. Then x_1 and x_2 can be represented by continuous maps of S^2 whose images have $|r|$ points in common. In particular, if $x_1 \cdot x_2 = 0$, x_1 and x_2 can be separated.*

3. The key lemma. Let S_1, S_2, S_3 be smoothly immersed generic 2-spheres in M such that their mutual algebraic intersection numbers are all zero. We denote by $p_\lambda^{(i,j)}$ (or $q_\lambda^{(i,j)}$) the λth positive (or negative) intersection point of S_i and S_j. Draw a smoothly imbedded arc $\gamma_{\lambda,i}^{(i,j)}$ (or $\gamma_{\lambda,j}^{(i,j)}$) connecting $p_\lambda^{(i,j)}$ and $q_\lambda^{(i,j)}$ on the immersed sphere S_i (or S_j). We assume that $\gamma_{\lambda,i}^{(i,j)} \cap \gamma_{\mu,i}^{(i,k)} = \varnothing$ ($\lambda \neq \mu$ or $j \neq k$). Let $\Delta_\lambda^{(i,j)}$ be a smoothly immersed generic 2-disk bounding the circle $\gamma_{\lambda,i}^{(i,j)} \cup \gamma_{\lambda,j}^{(i,j)}$.

LEMMA. *Suppose that $\Delta_1^{(i,j)} \cap S_k = \{a_1, \ldots, a_m\}$ and $\Delta_2^{(i,j)} \cap S_k = \{b_1, \ldots, b_n\}$ where $\{i, j, k\} = \{1, 2, 3\}$. Then one can regularly homotope S_1, S_2, S_3 to obtain S_1', S_2', S_3' and Whitney disks $\{\Delta_\lambda'^{(i,j)}\}$ such that:*

(1) $S_i' \cap S_j' = S_i \cap S_j$ ($\forall i, j$),

(2) $\Delta_\lambda'^{(i',j')} \cap S_{k'}' = \Delta_\lambda^{(i',j')} \cap S_k$, *for $\lambda > 2$, and*

(3) $\Delta_1'^{(i,j)} \cap S_k' = \{a_2, \ldots, a_m\}$, $\Delta_2'^{(i,j)} \cap S_k' = \{b_0, b_1, \ldots, b_n\}$ *where b_0 and a_1 have the same sign.*

PROOF. (See Figure 2.) Make S_k escape from the intersection point a_1 with $\Delta_1^{(i,j)}$ across $\gamma_{1,i}^{(i,j)}$, adding new intersections of S_k and S_i, $p_0^{(k,i)}$ and $q_0^{(k,i)}$; then we obtain a small Whitney disk Δ'. Choose an imbedding $g: B = [-1, 1] \times [0, 1] \to S_i$ such that

$$g([-1, 1] \times \{0\}) = \partial\Delta' \cap S_i, \qquad g(\{1\} \times \{0\}) = p_0^{\{k,i\}},$$

$$g(\{-1\} \times \{0\}) = q_0^{\{k,i\}}, \qquad g(B) \cap \gamma_{\lambda,i}^{\{i,l\}} = \varnothing$$

except for $l = j$ and $\lambda = 1$ or 2, and

$$g(B) \cap \gamma_{1,i}^{\{i,j\}} = \gamma_{1,i}^{\{i,j\}}\left(\left[0, \tfrac{1}{2}\right]\right) = g_1, \qquad g(B) \cap \gamma_{2,i}^{\{i,j\}} = \gamma_{2,i}^{\{i,j\}}\left(\left[\tfrac{1}{2}, 1\right]\right) = g_2,$$

where

$$\gamma_{1,i}^{\{i,j\}}(0) = p_1^{\{i,j\}}, \qquad \gamma_{2,i}^{\{i,j\}}(1) = q_2^{\{i,j\}}.$$

Let $\gamma_{0,k}^{\{k,i\}} = \partial\Delta' \cap S_k$,

$$\gamma_{0,i}^{\{k,i\}} = g(\{1, -1\} \times [0, 1] \cup [-1, 1] \times \{1\}),$$

and $\tilde{\tilde{\Delta}}_0^{\{k,i\}} = g(B) \cup \Delta'$. Then $\tilde{\tilde{\Delta}}_0^{\{k,i\}} \cap S_j = \{p_1^{\{i,j\}}, q_2^{\{i,j\}}\}$. Let ψ be a vector field on the arc $g(\{0\} \times [0, 1])$ which does not lie in $T(S_i)$, $T(S_i)| g_l +$ $T(\Delta^{\{i,j\}})| g_l$ ($l = 1, 2$). Push $\tilde{\tilde{\Delta}}_0^{\{k,i\}}$ off S_i along ψ keeping $\gamma_{0,i}^{\{k,i\}}$ fixed; then we obtain an imbedded disk $\tilde{\Delta}_0^{\{k,i\}}$ bounded by $\gamma_0^{\{k,i\}} \cup \gamma_{0,k}^{\{k,i\}}$ such that it meets S_i normally along $\gamma_{0,i}^{\{k,i\}}$, S_k normally along $\gamma_{0,k}^{\{k,i\}}$, and

$$S_i \cap \text{int } \tilde{\Delta}_0^{\{k,i\}} = \varnothing, \qquad S_k \cap \text{int } \tilde{\Delta}_0^{\{k,i\}} = \varnothing,$$

$$S_j \cap \tilde{\Delta}_0^{\{k,i\}} = \{p, q\}, \qquad \tilde{\Delta}_0^{\{k,i\}} \cap \Delta_\lambda^{\{i,j\}} = \varnothing$$

for $\lambda = 1, 2$. We can cancel these intersection points p and q as follows. Let γ (or γ') be a generic arc connecting p and q on $\tilde{\Delta}_0^{\{k,i\}}$ (or S_j), and let Δ be a good generic immersed disk bounded by $\gamma \cup \gamma'$. We can make S_j escape from the intersection with int Δ across γ'. Doing Whitney's trick for $\tilde{\Delta}_0^{\{k,i\}}$ and S_j across γ', we obtain a new immersed disk $\Delta_0^{\{k,i\}}$ such that $S_j \cap \Delta_0^{\{k,i\}} = \varnothing$. We may assume that $\Delta_0^{\{k,i\}}$ is good, and

$$\Delta_0^{\{k,i\}} \cap \Delta_\lambda^{\{j,k\}} = \varnothing, \qquad \text{int } \Delta_0^{\{k,i\}} \cap \text{int } \Delta_\mu^{\{i,j\}} = \varnothing.$$

For example, if $\Delta_0^{\{k,i\}} \cap \Delta_\lambda^{\{j,k\}} \neq \varnothing$, we can make $\Delta_0^{\{k,i\}}$ escape from the intersection with $\Delta_\lambda^{\{j,k\}}$ across $\gamma_{\lambda,k}^{\{j,k\}}$ by adding two intersection points of $\Delta_0^{\{k,i\}}$ and S_k. Possibly int $\Delta_0^{\{k,i\}} \cap S_k \neq \varnothing$, int $\Delta_0^{\{k,i\}} \cap S_i \neq \varnothing$. Make $\Delta_0^{\{k,i\}}$ escape from this intersection with S_k across $\gamma_{0,k}^{\{k,i\}}$, if necessary.

Using $\Delta_0^{\{k,i\}}$, we can do Whitney's trick for S_k and S_i across $\gamma_{0,k}^{\{k,i\}}$ and we obtain a new immersed 2-sphere S_i' such that

$$S_i' \cap S_k = S_i \cap S_k - \{p_0^{\{k,i\}}, q_0^{\{k,i\}}\} \quad \text{and} \quad \Delta_\lambda^{\{j,k\}} \cap S_i' = \Delta_\lambda^{\{j,k\}} \cap S_i.$$

Let $\tilde{f}$ denote the immersion: $D' \times [-1, 1] \to M$ in Device 3 such that $\tilde{f}(C_2 \times \{0\}) = \gamma_{0,i}^{\{k,i\}}$. We may assume that

$$\tilde{f}(\{(0, 0)\} \times [-1, 1]) = \gamma_{2,i}^{\{i,j\}} \cap \tilde{f}(D' \times [-1, 1]).$$

We shall modify the disk $\Delta_2^{\{i,j\}}$ as follows:

$$\Delta_2'^{\{i,j\}} = \Delta_2^{\{i,j\}} \cup \tilde{f}(\{(x, 0); 0 < x < 6/5\} \times [-1, 1]).$$
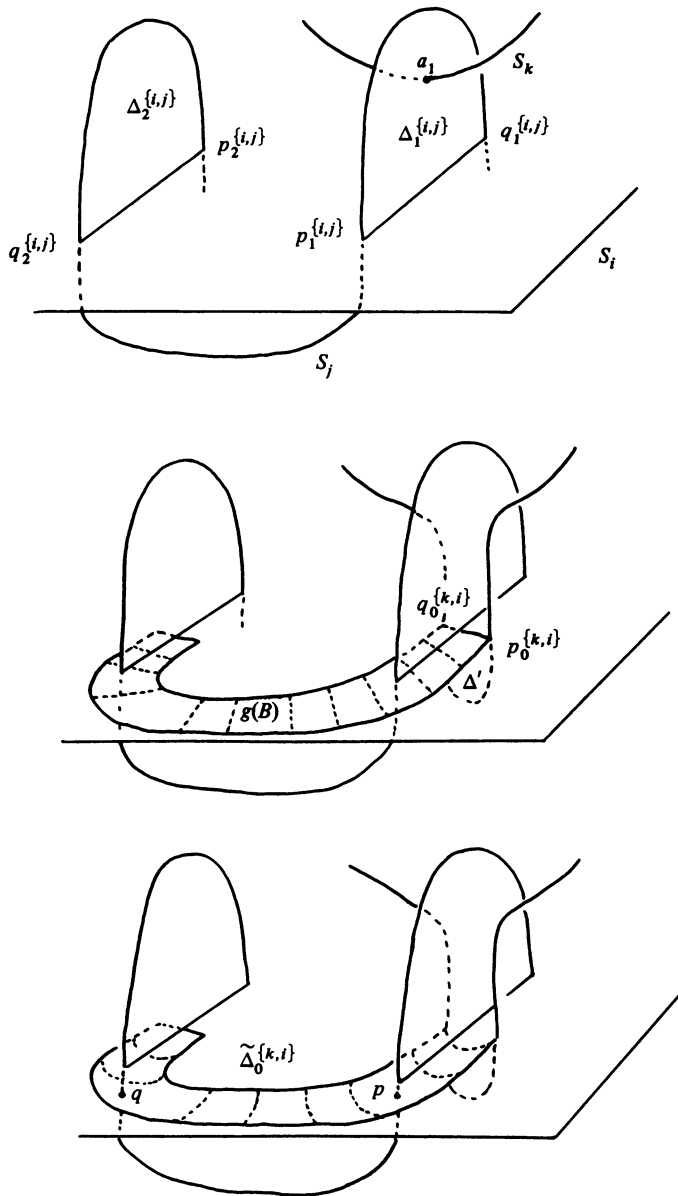
Then we obtain a new intersection point b_0.   Q.E.D.

FIGURE 2

4. Proof of theorem and corollary. Let x_1, x_2, $x_3 \in H_2(M; Z)$ be homology classes such that $x_i \cdot x_j = 0$ for $i \neq j$. Represent x_1, x_2, x_3 by smoothly immersed generic 2-spheres S_1, S_2, S_3, and let $p_\lambda^{(i,j)}$, $\gamma_{\lambda,i}^{(i,j)}$, $\Delta_\lambda^{(i,j)}$ be as in §3, but we do not require the condition $\gamma_{\lambda,i}^{(i,j)} \cap \gamma_{\mu,i}^{(i,k)} = \varnothing$ ($\lambda \neq \mu$ or $j \neq k$). The Whitney disk $\Delta_\lambda^{(i,j)}$ is oriented as in Figure 3. Now the Matsumoto triple $\langle x_1, x_2, x_3 \rangle$ is defined as follows:

$$\langle x_1, x_2, x_3 \rangle = \sum_\lambda S_1 \cdot \Delta_\lambda^{\{2,3\}} + \sum_\mu S_2 \cdot \Delta_\mu^{\{3,1\}} + \sum_\nu S_3 \cdot \Delta_\nu^{\{1,2\}}$$

$$+ \sum_{\mu,\nu} \frac{\partial \Delta_\mu^{\{3,1\}} \cdot \partial \Delta_\nu^{\{1,2\}}}{S_1} + \sum_{\nu,\lambda} \frac{\partial \Delta_\nu^{\{1,2\}} \cdot \partial \Delta_\lambda^{\{2,3\}}}{S_2}$$

$$+ \sum_{\lambda,\mu} \frac{\partial \Delta_\lambda^{\{2,3\}} \cdot \partial \Delta_\mu^{\{3,1\}}}{S_3} \quad \mod I,$$

where $S_1 \cdot \Delta_\lambda^{\{2,3\}}$, etc., denote the intersection number of S_1 and $\Delta_\lambda^{\{2,3\}}$, etc., and $(\partial \Delta_\mu^{\{3,1\}} \cdot \partial \Delta_\nu^{\{1,2\}})/S_1$, etc. denote the intersection number of $\partial \Delta_\mu^{\{3,1\}}$ and $\partial \Delta_\nu^{\{1,2\}}$ on S_1, etc.

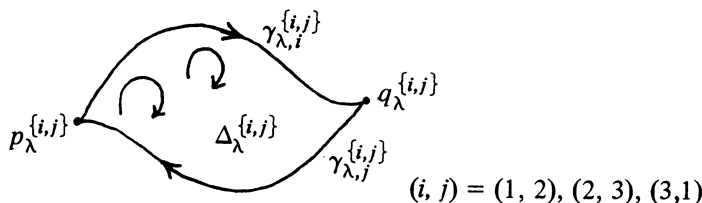$$(i, j) = (1, 2), (2, 3), (3,1)$$

FIGURE 3

PROOF OF THEOREM. Let $x_1, x_2, x_3 \in H_2(M; Z)$ be homology classes such that $x_i \cdot x_j = 0$ for $i \neq j$ and $\langle x_1, x_2, x_3 \rangle = 0 \in Z/I$. Let S_i, $\gamma_{\lambda,i}^{\{i,j\}}$, $\Delta_\lambda^{\{i,j\}}$ be as above. Now we assume as in §3 that $\gamma_{\lambda,i}^{\{i,j\}} \cap \gamma_{\mu,i}^{\{i,k\}} = \varnothing$ ($\lambda \neq \mu$ or $j \neq k$); then the Matsumoto triple $\langle x_1, x_2, x_3 \rangle$ is defined as

$$\sum_\lambda S_1 \cdot \Delta_\lambda^{\{2,3\}} + \sum_\mu S_2 \cdot \Delta_\mu^{\{3,1\}} + \sum_\nu S_3 \cdot \Delta_\nu^{\{1,2\}}.$$

We may assume that this sum is zero. In fact, if the ideal I is $\{0\}$, it is always zero. If I is not $\{0\}$, there exist homology classes $y_1, y_2, y_3 \in H_2(M; Z)$ such that $\langle x_1, x_2, x_3 \rangle = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$. Let F_1, F_2, F_3 be immersed 2-spheres representing y_1, y_2, y_3. Make connected-sums of $\Delta_1^{\{2,3\}}$ and $-F_1$, $\Delta_1^{\{3,1\}}$ and $-F_2$, $\Delta_1^{\{1,2\}}$ and $-F_3$, where $-F$ is an immersed 2-sphere with the reversed orientation. Then if we use the resulting immersed disks instead of $\Delta_1^{\{2,3\}}$, $\Delta_1^{\{3,1\}}$, $\Delta_1^{\{1,2\}}$, the sum is zero. We may assume that every Whitney disk is good in the sense of §2 and that there is no mutual-intersection of Whitney disks (and even there is no self-intersection of Whitney disks, i.e. every Whitney disk is an imbedded disk). (See proof of lemma.) As $x_1 \cdot x_3 = 0$, we may assume $S_1 \cap S_3 = \varnothing$ by the proposition. Escaping the intersection $S_1 \cap \text{int } \Delta_\lambda^{\{1,2\}}$ across $\gamma_{\lambda,1}^{\{1,2\}}$, we obtain a new immersed 2-sphere S_1', so that $S_1' \cap \text{int } \Delta_\lambda^{\{1,2\}} = \varnothing$ and $S_1' \cap S_3 = \varnothing$. Using $\Delta_\lambda^{\{1,2\}}$, do Whitney's trick for S_1' and S_2 across $\gamma_{\lambda,1}^{\{1,2\}}$, and we shall obtain a new immersed 2-sphere S_2' such that $S_1' \cap S_2' = \varnothing$. By Lemma, we may assume that $S_1' \cap \Delta_\lambda^{\{2,3\}} = \varnothing$ for $\lambda \neq 1$. Then $S_1' \cdot \Delta_1^{\{2,3\}} = 0$. Using Devices 1, 2 and 3, we obtain $S_1' \cap \Delta^{\{2,3\}} = \varnothing$. Now we can do Whitney's trick for S_2' and S_3 (using Device 2), and we obtain the required maps.   Q.E.D.

PROOF OF COROLLARY. If one of x_1, x_2, x_3 is 0, then this follows immediately from the proposition. If one of x_1, x_2, x_3, say x_1, is a primitive element, i.e. there is no homology class $x \in H_2(M; Z)$ such that $x_1 = mx$ ($m \in Z, \neq 1, -1$), then there exists a homology class $y \in H_2(M; Z)$ such that $x_1 \cdot y = 1$. Therefore $\langle x_1, x_2, x_3 \rangle = 0 \mod I = (1)$ and they can be separated by the theorem. If x_1, x_2, x_3 can be separated, also mx_1, x_2, x_3 can be separated by using the "self-connected-sum" of the immersed 2-sphere representing x_i ($m \in Z$).   Q.E.D.

REFERENCES

1. M. Freedman and R. Kirby, *A geometric proof of Rochlin's theorem*, Proc. Sympos. Pure Math., vol. 32, Part 2, Amer. Math. Soc., Providence, R. I., 1978, pp. 85–98.

2. K. Kobayashi, *On a homotopy version of 4-dimensional Whitney's lemma*, Math. Seminar Notes Kobe Univ. 5 (1977), 109–116.

3. Y. Matsumoto, *Secondary intersectional properties of 4-manifolds and Whitney's trick*, Proc. Sympos. Pure Math., vol. 32, Part 2, Amer. Math. Soc., Providence, R. I., 1978, pp. 99–107.

4. J. Milnor, *Lectures on the h-cobordism theorem*, Math. Notes, vol. 1, Princeton Univ. Press, Princeton, N. J., 1965.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF TOKYO, HONGO, TOKYO, 113, JAPAN