WHITNEY’S TRICK FOR THREE 2-DIMENSIONAL HOMOLOGY CLASSES OF 4-MANIFOLDS

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Abstract. In his recent paper, Y. Matsumoto has defined a triple product of 2-homology classes of simply-connected oriented 4-manifolds, when the intersection numbers are zero. In the present paper, the author establishes that three 2-homology classes can be homotopically separated if the intersection numbers and the triple product vanish.

1. Introduction. Let $M$ be a simply-connected oriented 4-manifold possibly with boundary. We shall say that homology classes $x_i \in H_2(M; \mathbb{Z})$, $i = 1, \ldots, n$, can be separated, if there exist continuous maps $f_i: S^2 \to M$ representing $x_i$, $i = 1, \ldots, n$, such that $f_i(S^2) \cap f_j(S^2) = \emptyset$ for $i \neq j$.

In [2], establishing a homotopy version of Whitney’s trick in dimension 4, Kobayashi proved that homology classes $x_1$ and $x_2 \in H_2(M; \mathbb{Z})$ can be separated if and only if the intersection number $x_1 \cdot x_2 = 0$.

On the other hand, Matsumoto [3] defined a “secondary intersection triple”, which we shall call Matsumoto triple, $\langle x_1, x_2, x_3 \rangle \in \mathbb{Z}/I$, where $x_1, x_2, x_3$ are 2-dimensional homology classes of $M$ such that $x_i \cdot x_j = 0$ for $i \neq j$ and $I$ is an ideal of $\mathbb{Z}$, $\{ x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3 \in \mathbb{Z}; y_1, y_2, y_3 \in H_2(M, \mathbb{Z}) \}$. See §4 for the definition. He showed that $x_1, x_2, x_3$ cannot be separated if the triple $\langle x_1, x_2, x_3 \rangle \neq 0$.

In this paper we shall prove the following:

Theorem. Let $M$ be a simply-connected oriented 4-manifold, then three homology classes $x_1, x_2, x_3 \in H_2(M; \mathbb{Z})$ can be separated if and only if the intersection numbers $x_i \cdot x_j = 0$ for $i \neq j$ and the Matsumoto triple $\langle x_1, x_2, x_3 \rangle = 0$.

Corollary. When $M$ is closed, homology classes $x_1, x_2, x_3$ can be separated if and only if the intersection numbers $x_i \cdot x_j = 0$ for $i \neq j$.

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2. Some fundamental devices. Throughout this paper, we shall denote by $M$ a simply-connected oriented 4-manifold. Let $f_1, f_2: S^2 \to M$ be smooth generic immersions in the sense that all the self- and mutual-intersections of
$S_1(= f_1(S^2))$ and $S_2(= f_2(S^2))$ are transversal double points. Suppose that $S_1 \cap S_2$ consists of positive double points, say $p_1, \ldots, p_m$, and negative ones, say $q_1, \ldots, q_n$, where $m, n \neq 0$. Draw smooth arcs $\gamma_1, \gamma_2$ connecting $p_1$ and $q_1$ on $S_1, S_2$, respectively. We may assume that $\gamma_1$ and $\gamma_2$ are generic in the sense that they have no self-intersection points and pass through neither the self-intersection points nor the mutual intersection points of $S_1$ and $S_2$. As $M$ is simply-connected, $\gamma_1 \cup \gamma_2$ bounds a smoothly immersed 2-disk $\Delta$, called a Whitney disk, which is generic with respect to $S_1$ and $S_2$. Let $\phi$ be a nonzero vector field on $\gamma_1 \cup \gamma_2$ such that when restricted to $\gamma_2$ it gives a cross-section of a normal 1-vector bundle $\nu(\gamma_2 \to S_2)$ and when restricted to $\gamma_1$ the unique extension of $\phi|_{\{p_1,q_1\}}$ over $\gamma_1$ which is normal to both $S_1$ and $\Delta$. Let $\Theta(\Delta) \in \mathbb{Z} = \pi_1(SO_2)$ be the obstruction to extending $\phi$ over $\Delta$. We shall say that the Whitney disk $\Delta$ is good, if $\Theta(\Delta) = 0$.

Device 1 (Making the Whitney disk $\Delta$ good). When the Whitney disk $\Delta$ is not good, one can obtain a good Whitney disk spanning $\gamma_1 \cup \gamma_2$ by spinning $\Delta$ around $\gamma_1$ or $\gamma_2$ (see [1]).

Device 2 (Making $S_1$ escape from the intersection with $\text{int} \Delta$ across $\gamma_1$). If $S_1 \cap \text{int} \Delta \neq \emptyset$, for a point $p \in S_1 \cap \text{int} \Delta$ we take a point $p' \in \text{int} \gamma_1$ and a simple arc $\gamma$ connecting $p$ and $p'$ on $\Delta$ such that $\gamma \cap S_1 = \{p, p'\}$ and $\gamma \cap S_2 = \emptyset$. Pushing a neighborhood of the intersecting point $p$ in $S_1$ along the arc $\gamma$ off $\Delta$ as in Figure 1, we can make $S_1$ escape from the intersection point $p$ with $\text{int} \Delta$ across $\gamma_1$, by adding two self-intersection points with opposite sign for $S_1$.

Device 3 (Whitney's trick for $S_1$ and $S_2$ across $\gamma_1$). Let $\Delta$ be a good immersed Whitney disk such that $S_1 \cap \text{int} \Delta = \emptyset$. We set

$$D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1, y > 0\},$$

$$D' = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < (6/5)^2, y > 0\},$$

$$C_1 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1, y > 0\},$$

$$C_1' = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = (6/5)^2, y > 0\}.$$

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Let \( f: D \to M \) be an immersion such that \( f(D) = \Delta, f(C_i) = \gamma_1,f(C_j) = \gamma_2 \). Adding a collar along \( \gamma_1 \), we have an extension \( f': D' \to M \) of \( f \), so that \( f'(C_i) \cap S_1 = \emptyset, f'(C_j) \subset S_2 \). Since \( \Delta \) is good, we have an immersion \( \tilde{f}: D' \times [-1,1] \to M \) using the vector field \( \tilde{\phi} \) (the extension of \( \phi \) over \( \Delta \)) such that \( \tilde{f} \) restricted to \( D' = D' \times \{0\} \) coincides with \( f' \). Now the immersed 2-sphere \( S_2 \) shall be modified as follows.

\[
S'_2 = \left(S_2 - \tilde{f}(C'_i \times [-1,1])\right) \cup \tilde{f}(C'_i \times [-1,1]) \cup \tilde{f}(D' \times \{-1,1\}).
\]

Rounding the corners, one can assume \( S'_2 \) is a generic immersed 2-sphere.

Now \( S_1 \cap S'_2 = \{p_2, \ldots , p_m, q_2, \ldots , q_n\} \). It is easy to construct a generic immersion \( f_2: S^2 \to M \) with \( f_2(S^2) = S'_2 \) which is regularly homotopic to \( f_2 \). This process will be referred to as Whitney's trick for \( S_1 \) and \( S_2 \) across \( \gamma_1 \) (see [4, Theorem 6.6]).

**Remark.** (1) Let \( X \) be a compact subset of \( M \). If \( \Delta \cap X = \emptyset \), then \( S_2 \cap X = S_2 \cap X \) in Device 3.

(2) These can be applied to generic intersections of immersed disks and spheres not only of immersed spheres.

Using the Devices 1, 2 and 3 repeatedly, we obtain

**Proposition (Kobayashi [2]).** Let \( x_1, x_2 \in H_2(M; \mathbb{Z}) \) be homology classes such that \( x_1 \cdot x_2 = r \). Then \( x_1 \) and \( x_2 \) can be represented by continuous maps of \( S^2 \) whose images have \(|r|\) points in common. In particular, if \( x_1 \cdot x_2 = 0 \), \( x_1 \) and \( x_2 \) can be separated.

3. The key lemma. Let \( S_1, S_2, S_3 \) be smoothly immersed generic 2-spheres in \( M \) such that their mutual algebraic intersection numbers are all zero. We denote by \( p_{\lambda}(u) \) (or \( q_{\lambda}(u) \)) the \( \lambda \)th positive (or negative) intersection point of \( S_i \) and \( S_j \). Draw a smoothly imbedded arc \( \gamma_{\lambda}(u) \) (or \( \gamma_{\lambda}(u) \)) connecting \( p_{\lambda}(u) \) and \( q_{\lambda}(u) \) on the immersed sphere \( S_i \) (or \( S_j \)). We assume that \( \gamma_{\lambda}(u) \cap \gamma_{\mu}(v) = \emptyset \) (\( \lambda \neq \mu \) or \( j \neq k \)). Let \( \Delta_{\lambda}(u) \) be a smoothly immersed generic 2-disk bounding the circle \( \gamma_{\lambda}(u) \cup \gamma_{\lambda}(v) \).

**Lemma.** Suppose that \( \Delta_{\lambda}(u) \cap S_k = \{a_1, \ldots , a_n\} \) and \( \Delta_{\lambda}(v) \cap S_k = \{b_1, \ldots , b_n\} \) where \( \{i,j,k\} = \{1,2,3\} \). Then one can regularly homotope \( S_1, S_2, S_3 \) to obtain \( S'_1, S'_2, S'_3 \) and Whitney disks \( \{\Delta_{\lambda}^{(i)}(u)\} \) such that:

1. \( S'_i \cap S'_j = S_i \cap S_j \) \( \forall i,j \),
2. \( \Delta_{\lambda}^{(i)}(u) \cap S'_k = \Delta_{\lambda}^{(i)}(u) \cap S_k \) for \( \lambda > 2 \), and
3. \( \Delta_{\lambda}^{(i)}(u) \cap S'_k = \{a_2, \ldots , a_m\}, \Delta_{\lambda}^{(i)}(v) \cap S'_k = \{b_0, b_1, \ldots , b_n\} \) where \( b_0 \) and \( a_1 \) have the same sign.

**Proof.** (See Figure 2.) Make \( S_k \) escape from the intersection point \( a_k \) with \( \Delta_i^{(i)} \) across \( \gamma_{\lambda}(u) \), adding new intersections of \( S_k \) and \( p_{0}(u) \) and \( q_{0}(v) \); then we obtain a small Whitney disk \( \Delta' \). Choose an imbedding \( g: B = [-1,1] \times [0,1] \to S_i \) such that
\[ g([-1, 1] \times \{0\}) = \partial \Delta' \cap S, \quad g(\{1\} \times \{0\}) = p_i^{(k,i)}, \]
\[ g((-1) \times \{0\}) = q_0^{(k,i)}, \quad g(B) \cap \gamma_i^{(k,i)} = \emptyset \]

except for \( i = j \) and \( \lambda = 1 \) or \( 2 \), and
\[ g(B) \cap \gamma_i^{(j,i)} = \gamma_i^{(j,i)}([0, \frac{1}{2}]) = g_1, \quad g(B) \cap \gamma_i^{(j,i)} = \gamma_i^{(j,i)}([\frac{1}{2}, 1]) = g_2, \]

where
\[ \gamma_1^{(j,i)}(0) = p_i^{(j,i)}, \quad \gamma_2^{(j,i)}(1) = q_i^{(j,i)}. \]

Let \( \gamma_0^{(k,i)} = \partial \Delta' \cap S_k \),
\[ \gamma_0^{(k,i)} = g((-1) \times [0, 1] \cup [-1, 1] \times \{1\}), \]

and \( \tilde{\Delta}_0^{(k,i)} = g(B) \cup \Delta' \). Then \( \tilde{\Delta}_0^{(k,i)} \cap S_j = \{p_i^{(j,i)}, q_2^{(j,i)}\} \). Let \( \psi \) be a vector field on the arc \( g((0) \times [0, 1]) \) which does not lie in \( T(S_j), T(S_j) \cap g_l + T(\Delta_j^{(l,i)}) \cap g_l \) \((l = 1, 2)\). Push \( \tilde{\Delta}_0^{(k,i)} \) off \( S_j \) along \( \psi \) keeping \( \gamma_0^{(k,i)} \) fixed; then we obtain an imbedded disk \( \tilde{\Delta}_0^{(k,i)} \) bounded by \( \gamma_0^{(k,i)} \cup \gamma_0^{(k,i)} \) such that it meets \( S_j \) normally along \( \gamma_0^{(k,i)} \), \( S_k \) normally along \( \gamma_0^{(k,i)} \), and
\[ S_j \cap \text{int } \Delta_0^{(k,i)} = \emptyset, \quad S_k \cap \text{int } \Delta_0^{(k,i)} = \emptyset, \]
\[ S_j \cap \Delta_0^{(k,i)} = \{p, q\}, \quad \Delta_0^{(k,i)} \cap \Delta_j^{(k,i)} = \emptyset \]

for \( \lambda = 1, 2 \). We can cancel these intersection points \( p \) and \( q \) as follows. Let \( \gamma \) (or \( \gamma' \)) be a generic arc connecting \( p \) and \( q \) on \( \tilde{\Delta}_0^{(k,i)} \) (or \( S_j \)), and let \( \Delta \) be a good generic immersed disk bounded by \( \gamma \cup \gamma' \). We can make \( S_j \) escape from the intersection with \( \text{int } \Delta \) across \( \gamma' \). Doing Whitney’s trick for \( \tilde{\Delta}_0^{(k,i)} \) and \( S_j \) across \( \gamma' \), we obtain a new immersed disk \( \Delta_0^{(k,i)} \) such that \( S_j \cap \Delta_0^{(k,i)} = \emptyset \). We may assume that \( \Delta_0^{(k,i)} \) is good, and
\[ \Delta_0^{(k,i)} \cap \Delta_j^{(k,i)} = \emptyset, \quad \text{int } \Delta_0^{(k,i)} \cap \text{int } \Delta_j^{(k,i)} = \emptyset. \]

For example, if \( \Delta_0^{(k,i)} \cap \Delta_j^{(k,i)} \neq \emptyset \), we can make \( \Delta_0^{(k,i)} \) escape from the intersection with \( \Delta_j^{(k,i)} \) across \( \gamma_0^{(k,i)} \) by adding two intersection points of \( \Delta_0^{(k,i)} \) and \( S_k \). Possibly \( \text{int } \Delta_0^{(k,i)} \cap S_k = \emptyset \), \( \text{int } \Delta_0^{(k,i)} \cap S_k = \emptyset \). Make \( \Delta_0^{(k,i)} \) escape from this intersection with \( S_k \) across \( \gamma_0^{(k,i)} \), if necessary.

Using \( \Delta_0^{(k,i)} \), we can do Whitney’s trick for \( S_k \) and \( S_j \) across \( \gamma_0^{(k,i)} \) and we obtain a new immersed 2-sphere \( S' \) such that
\[ S'_j \cap S_k = S_j \cap S_k = \{p_0^{(k,i)}, q_0^{(k,i)}\} \quad \text{and} \quad \Delta_0^{(k,i)} \cap S'_j = \Delta_0^{(k,i)} \cap S_j. \]

Let \( \tilde{f} \) denote the immersion: \( D' \times [-1, 1] \to M \) in Device 3 such that \( \tilde{f}(C_x \times \{0\}) = \gamma_0^{(k,i)} \). We may assume that
\[ \tilde{f}([-1, 1] \times (0, 0)) = \gamma_2^{(j,i)} \cap \tilde{f}(D' \times [-1, 1]). \]

We shall modify the disk \( \Delta_2^{(j,i)} \) as follows:
\[ \Delta_2^{(j,i)} = \Delta_2^{(j,i)} \cup \tilde{f}((-1, 1) \times [0 < x < 6/5] \times [-1, 1]). \]

Then we obtain a new intersection point \( b_0 \). Q.E.D.
4. Proof of theorem and corollary. Let $x_1, x_2, x_3 \in H_2(M; \mathbb{Z})$ be homology classes such that $x_i \cdot x_j = 0$ for $i \neq j$. Represent $x_1, x_2, x_3$ by smoothly immersed generic 2-spheres $S_1, S_2, S_3$, and let $p_{ij}^{(l)}, \gamma_{ij}^{(k)}, \Delta_{ij}^{(d)}$ be as in §3, but we do not require the condition $\gamma_{k,l}^{(d)} \cap \gamma_{\mu,l}^{(d)} = \emptyset$ ($\lambda \neq \mu$ or $j \neq k$). The Whitney disk $\Delta_{ij}^{(d)}$ is oriented as in Figure 3. Now the Matsumoto triple $\langle x_1, x_2, x_3 \rangle$ is defined as follows:
\[ \langle x_1, x_2, x_3 \rangle = \sum_{\lambda} S_{\lambda} \cdot \Delta^{(2,3)}_{\lambda} + \sum_{\mu} S_{2} \cdot \Delta^{(3,1)}_{\mu} + \sum_{\nu} S_{3} \cdot \Delta^{(1,2)}_{\nu} + \sum_{\mu, \nu} \frac{\partial \Delta^{(3,1)}_{\mu}}{S_{\lambda}} \cdot \frac{\partial \Delta^{(1,2)}_{\nu}}{S_{2}} + \sum_{\lambda, \mu} \frac{\partial \Delta^{(2,3)}_{\lambda}}{S_{3}} \cdot \frac{\partial \Delta^{(3,1)}_{\mu}}{S_{\lambda}} \text{ mod } I, \]

where \( S_{\lambda} \cdot \Delta^{(2,3)}_{\lambda} \), etc., denote the intersection number of \( S_{\lambda} \) and \( \Delta^{(2,3)}_{\lambda} \), etc., and \( (\partial \Delta^{(3,1)}_{\mu}/S_{\lambda}) \cdot \Delta^{(1,2)}_{\nu}/S_{2} \), etc. denote the intersection number of \( \partial \Delta^{(3,1)}_{\mu} \) and \( \partial \Delta^{(1,2)}_{\nu} \) on \( S_{\lambda} \), etc.

\[ \gamma_{\lambda,i}^{(i,j)} \]

\[ p_{\lambda}^{(i,j)} \]

\[ \Delta^{(i,j)}_{\lambda} \]

\[ q_{\lambda}^{(i,j)} \]

\( (i, j) = (1, 2), (2, 3), (3, 1) \)

**Figure 3**

**Proof of Theorem.** Let \( x_1, x_2, x_3 \in H_2(M; \mathbb{Z}) \) be homology classes such that \( x_i \cdot x_j = 0 \) for \( i \neq j \) and \( \langle x_1, x_2, x_3 \rangle = 0 \in \mathbb{Z}/I \). Let \( S_{\lambda}, \gamma_{\lambda,i}^{(i,j)}, \Delta^{(i,j)}_{\lambda} \) be as above. Now we assume as in §3 that \( \gamma_{\lambda,i}^{(i,j)} \cap \gamma_{\mu,j}^{(i,k)} = \emptyset \) (\( \lambda \neq \mu \) or \( j \neq k \)); then the Matsumoto triple \( \langle x_1, x_2, x_3 \rangle \) is defined as

\[ \sum_{\lambda} S_{\lambda} \cdot \Delta^{(2,3)}_{\lambda} + \sum_{\mu} S_{2} \cdot \Delta^{(3,1)}_{\mu} + \sum_{\nu} S_{3} \cdot \Delta^{(1,2)}_{\nu}. \]

We may assume that this sum is zero. In fact, if the ideal \( I \) is \( \{0\} \), it is always zero. If \( I \) is not \( \{0\} \), there exist homology classes \( y_1, y_2, y_3 \in H_2(M; \mathbb{Z}) \) such that \( \langle x_1, x_2, x_3 \rangle = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3 \). Let \( F_1, F_2, F_3 \) be immersed 2-spheres representing \( y_1, y_2, y_3 \). Make connected-sums of \( \Delta^{(2,3)}_{i} \) and \( \Delta^{(3,1)}_{i} \) and \( \Delta^{(1,2)}_{i} \), where \( -F \) is an immersed 2-sphere with the reversed orientation. Then if we use the resulting immersed disks instead of \( \Delta^{(2,3)}_{i}, \Delta^{(3,1)}_{i}, \Delta^{(1,2)}_{i} \), the sum is zero. We may assume that every Whitney disk is good in the sense of §2 and that there is no mutual-intersection of Whitney disks (and even there is no self-intersection of Whitney disks, i.e. every Whitney disk is an imbedded disk). (See proof of lemma.) As \( x_1 \cdot x_3 = 0 \), we may assume \( S_{\lambda} \cap S_{2} = \emptyset \) by the proposition. Escaping the intersection \( S_{\lambda} \cap \text{int } \Delta^{(1,2)}_{i} \) across \( \gamma_{\lambda,i}^{(1,2)} \), we obtain a new immersed 2-sphere \( S'_{\lambda} \), so that \( S'_{\lambda} \cap \text{int } \Delta^{(1,2)}_{i} = \emptyset \) and \( S'_{\lambda} \cap S_{2} = \emptyset \). Using \( \Delta^{(1,2)}_{i} \), do Whitney's trick for \( S'_{\lambda} \) and \( S_{2} \) across \( \gamma_{\lambda,i}^{(1,2)} \), and we shall obtain a new immersed 2-sphere \( S''_{\lambda} \) such that \( S''_{\lambda} \cap S_{2} = \emptyset \). By Lemma, we may assume that \( S''_{\lambda} \cap \Delta^{(2,3)}_{i} = \emptyset \) for \( \lambda \neq 1 \). Then \( S''_{\lambda} \cdot \Delta^{(2,3)}_{i} = 0 \). Using Devices 1, 2 and 3, we obtain \( S''_{\lambda} \cap \Delta^{(2,3)}_{i} = \emptyset \). Now we can do Whitney's trick for \( S''_{\lambda} \) and \( S_{3} \) (using Device 2), and we obtain the required maps. Q.E.D.
Proof of Corollary. If one of $x_1, x_2, x_3$ is 0, then this follows immediately from the proposition. If one of $x_1, x_2, x_3$, say $x_1$, is a primitive element, i.e. there is no homology class $x \in H_2(M; \mathbb{Z})$ such that $x_1 = mx$ ($m \in \mathbb{Z}, \neq 1, -1$), then there exists a homology class $y \in H_2(M; \mathbb{Z})$ such that $x_1 \cdot y = 1$. Therefore $\langle x_1, x_2, x_3 \rangle = 0 \mod I = (1)$ and they can be separated by the theorem. If $x_1, x_2, x_3$ can be separated, also $mx_1, x_2, x_3$ can be separated by using the “self-connected-sum” of the immersed 2-sphere representing $x_i$ ($m \in \mathbb{Z}$). Q.E.D.

References


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