PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 76, Number 1, August 1979

THE ETALE HOMOTOPY TYPE OF VARIETIES OVER R

DAVID A. COX

ABSTRACT. For a variety X over Spec($\mathbf{R}$), the étale homotopy type of X is computed in terms of the action of complex conjugation on the complex points $X(\mathbf{C})$. This enables one to show that $X(\mathbf{R}) \neq \varnothing$ is equivalent to various conditions on the étale cohomology of X, and, when X is a smooth, geometrically connected, proper curve over Spec($\mathbf{R}$), to compute the étale cohomology. Finally, there is a negative result, showing that étale cohomology cannot be used to compute the topological degree of a mapping germ f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$.

Let X be a variety over $\mathbf{R}$ (i.e., a scheme of finite type over Spec($\mathbf{R}$)). The étale homotopy of $\overline{X} = X \times_{\mathbf{R}} \mathbf{C}$, denoted $\{\overline{X}\}_{\text{et}}$, is well understood: there is a canonical weak homotopy equivalence:

$$\{\overline{X}\}\hat{}_{\text{et}} \simeq \overline{X}(\mathbf{C})\hat{}$$

where $\hat{}$ denotes profinite completion, and $X(\mathbf{C})$ is the set of complex points of X with the "classical" topology (see [2]). The étale homotopy type $\{X\}_{\text{et}}$ is harder to understand: its relation to $X(\mathbf{R})$ is not at all direct. The key is to realize that $\overline{X} \to X$ is étale, so that $\{\overline{X}\}_{\text{et}} \to \{X\}_{\text{et}}$ is a covering space with group $G = \mathbf{Z}/2\mathbf{Z}$. Thus, $\{X\}\hat{}_{\text{et}}$ is the quotient of $X(\mathbf{C})\hat{}$ by a *free* G-action. The usual action of G on $X(\mathbf{C})$ (given by complex conjugation) need not be free (e.g., when $X(\mathbf{R}) \neq \varnothing$), but there is a well-known way to correct this: take the diagonal action of G on $X(\mathbf{C}) \times S^\infty$, where G acts on S^∞ by the antipodal map. The quotient is denoted $X(\mathbf{C})_G$ (see [4, VII]), and we will prove that there is a canonical weak equivalence:

$$\{X\}\hat{}_{\text{et}} \simeq X(\mathbf{C})\hat{}_G.$$

Using this, we prove some results (announced in [3]) of Artin and Verdier which relate the cohomological dimension of X to the existence of rational points over $\mathbf{R}$. Then we answer (negatively) a question raised by D. Eisenbud as to whether étale cohomology can be used to compute the topological degree of a finite map germ f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$.

We would like to thank M. Artin and T. Chang for useful conversations.

1. Here is our main result:

THEOREM 1.1. *For any variety X over $\mathbf{R}$, there is a canonical weak homotopy equivalence in Pro-$\mathcal{H}$:*

$$\{X\}\hat{}_{\text{et}} \simeq X(\mathbf{C})\hat{}_G.$$

Received by the editors September 17, 1978.
AMS (MOS) subject classifications (1970). Primary 14F20, 14F25, 14F35.

PROOF. S^∞ is a contractible principal G-space E_G with quotient $B_G = \mathbf{RP}^\infty$. Both of these spaces can be realized as simplicial sets. Then we can regard $X(\mathbf{C}) \times E_G$ as a simplicial space:

$$X(\mathbf{C}) \times G \leftleftarrows X(\mathbf{C}) \times G \times G \overset{\leftarrow}{\leftleftarrows} \ldots$$

(we omit the arrows for degeneracies). If we divide this by the diagonal action of G (where G acts on itself on the right), then we have $X(\mathbf{C})_G$ represented as a simplicial space:

$$X(\mathbf{C}) \leftleftarrows X(\mathbf{C}) \times G \ldots . \tag{1}$$

The map from $X(\mathbf{C}) \times E_G$ to (1) is defined by sending $(x, g_0, \ldots, g_n)$ to $(g_n x, g_0 g_1^{-1}, \ldots, g_{n-1} g_n^{-1})$; from this the boundaries and degeneracies of (1) are easy to determine.

Since $\bar{X}$ is a G-torseur over X, $\operatorname{cos} \bar{\mathrm{k}}_0^X(X)$ (formed out of fiber products of $\bar{X}$ over X) becomes a simplicial scheme:

$$\bar{X} \leftleftarrows \bar{X} \times G \overset{\leftarrow}{\leftleftarrows} \ldots . \tag{2}$$

It is easy to check that (1) is the simplicial space formed by the complex points of (2). Thus, the comparison theorem of [**5**, IV. 3] shows that we have a canonical weak equivalence $\{\operatorname{cos} \bar{\mathrm{k}}_0^X(X)\}_{\mathrm{et}}^\wedge \simeq X(\mathbf{C})_G^\wedge$. $\operatorname{cos} \bar{\mathrm{k}}_0^X(X)$ is a hypercovering of X, so that the natural map $\{\operatorname{cos} \bar{\mathrm{k}}_0^X(X)\}_{\mathrm{et}} \to \{X\}_{\mathrm{et}}$ is also a weak equivalence (see [**5**, IV. 1]). □

The idea that (1) should represent the étale homotopy type of X is due to M. Artin.

Let X_∞ denote the set of closed points of X. Because $X_\infty = X(\mathbf{C})/G$, X_∞ inherits a "classical" topology from $X(\mathbf{C})$, and $X(\mathbf{R})$ becomes a closed subset of X_∞. Theorem 1.1 and the exact sequence (1.3) of [**4**, VII] tell us how the étale cohomology of X relates to X_∞ and $X(\mathbf{R})$:

PROPOSITION 1.2. *There is a long exact sequence*:

$$\to H^q(X_\infty, X(\mathbf{R}); M) \to H^q_{\mathrm{et}}(X, M) \to H^q(X(\mathbf{R}) \times B_G, M) \to$$

for any finite group. M. □

This can be quite useful for computing étale cohomology. For example, let X be a connected smooth complete curve over $\mathbf{R}$ of genus g, and assume that $X(\mathbf{R}) \neq \varnothing$. Then $X(\mathbf{C})$ is a compact Riemann surface of genus g, $X(\mathbf{R})$ is a disjoint union of circles (say, N of them), and X_∞ is a 2-manifold-with-boundary. Working with $\mathbf{Z}/2\mathbf{Z}$ coefficients, we have

$$2\chi(X_\infty, X(\mathbf{R})) = 2\chi(X_\infty - X(\mathbf{R})) = \chi(X(\mathbf{C}) - X(\mathbf{R})) = \chi(X(\mathbf{C}))$$

(because $\chi(X(\mathbf{R})) = 0$). Thus, we get

$$H^1(X_\infty, X(\mathbf{R})) \simeq (\mathbf{Z}/2\mathbf{Z})^g$$

since $H^2(X_\infty, X(\mathbf{R})) \simeq \mathbf{Z}/2\mathbf{Z}$. Also the map $H^2(X_\infty, X(\mathbf{R})) \to H^2_{\mathrm{et}}(X, \mathbf{Z}/2\mathbf{Z})$ is zero because it factors through $H^2(X_\infty) = 0$. From Proposition 1.2 we get:

$$H^q_{\text{et}}(X, \mathbf{Z}/2\mathbf{Z}) \simeq \begin{cases} \mathbf{Z}/2\mathbf{Z}, & q = 0, \\ (\mathbf{Z}/2\mathbf{Z})^{N+g}, & q = 1, \\ (\mathbf{Z}/2\mathbf{Z})^{2N}, & q \geqslant 2. \end{cases}$$

2. We next prove:

THEOREM 2.1 (ARTIN-VERDIER [3]). *Let X be a variety of dimension n over* **R**. *The following are equivalent*:

1. $X(\mathbf{R}) = \varnothing$.
2. *There is a weak equivalence* $\{X\}^{\wedge}_{\text{et}} \simeq X^{\wedge}_{\infty}$.
3. $\text{cd}_2 X < \infty$.
4. $\text{cd}_2 X \leqslant 2n$.

PROOF. $\text{cd}_2 X$ means the cohomological dimension of X with respect to étale 2-torsion sheaves. Because X_∞ is a $2n$-dimensional CW complex, Proposition 1.2 says that we have an isomorphism:

$$H^q_{\text{et}}(X, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\sim} H^q(X(\mathbf{R}) \times B_G, \mathbf{Z}/2\mathbf{Z}) \tag{3}$$

for $q > 2n$. Thus $4 \Rightarrow 3 \Rightarrow 1$ and $2 \Rightarrow 1$ are immediate. If $X(\mathbf{R}) = \varnothing$, then G acts freely on $X(\mathbf{C})$, so that the natural map $X_\infty \to X(\mathbf{C})_G$ is a homotopy equivalence. Then $\{X\}^{\wedge}_{\text{et}} \simeq X(\mathbf{C})^{\wedge}_G \simeq X^{\wedge}_{\infty}$. Finally, we must show $1 \Rightarrow 4$. If Y is any scheme which is integral and finite over X, then $Y(\mathbf{R}) = \varnothing$ (since $X(\mathbf{R})$ is), so that by (3), $H^q_{\text{et}}(Y, \mathbf{Z}/2\mathbf{Z}) = 0$ for $q > 2 \dim Y$. Then Proposition 5.6 of [**1**, IX] (where the function ϕ referred to in the statement of that proposition is given here by $\phi(Y) = 2 \dim Y$) shows that $\text{cd}_2 X \leqslant 2n$. □

In some cases, there are other relations between rational points and étale cohomology. For example:

PROPOSITION 2.2. *Let X be smooth, proper and geometrically connected over* **R**. *If X has dimension $n > 0$, then the following are equivalent*:

1. $X(\mathbf{R}) = \varnothing$.
2. $H^{2n}_{\text{et}}(X, \mathbf{Z}/2\mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z}$.
3. *The map* $H^{2n}_{\text{et}}(X, \mathbf{Z}/2\mathbf{Z}) \to H^{2n}_{\text{et}}(\overline{X}, \mathbf{Z}/2\mathbf{Z})$ *is zero.*

PROOF. If $X(\mathbf{R}) = \varnothing$, then X_∞ is a connected compact $2n$-manifold. Thus

$$H^{2n}_{\text{et}}(X, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\sim} H^{2n}(X_\infty, \mathbf{Z}/2\mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z},$$

and the map $H^{2n}(X_\infty, \mathbf{Z}/2\mathbf{Z}) \to H^{2n}(X(\mathbf{C}), \mathbf{Z}/2\mathbf{Z})$ is zero because the map $X(\mathbf{C}) \to X_\infty$ has degree 2.

If $X(\mathbf{R}) \neq \varnothing$, then $X(\mathbf{R})$ is a disjoint union of compact n-manifolds. Thus $H^{2n}(X(\mathbf{R}) \times B_G, \mathbf{Z}/2\mathbf{Z})$ has at least two nonzero summands,

$$H^0(X(\mathbf{R}), \mathbf{Z}/2\mathbf{Z}) \otimes H^{2n}(B_G, \mathbf{Z}/2\mathbf{Z})$$

and

$$H^n(X(\mathbf{R}), \mathbf{Z}/2\mathbf{Z}) \otimes H^n(B_G, \mathbf{Z}/2\mathbf{Z}).$$

Since $H^{2n}_{\text{et}}(X, \mathbf{Z}/2\mathbf{Z}) \to H^{2n}(X(\mathbf{R}) \times B_G, \mathbf{Z}/2\mathbf{Z})$ is surjective by Proposition 1.2, $H^{2n}_{\text{et}}(X, \mathbf{Z}/2\mathbf{Z})$ has at least 4 elements.

A point $x \in X(\mathbf{R})$ gives a closed subscheme $Y = \{x\}$ of X of codimension n, and this gives us a cycle class $\text{cl}(Y) \in H^{2n}_Y(X, \mathbf{Z}/2\mathbf{Z})$ (see [**6**, Cycle 2.3]). If we set $\bar{Y} = Y \times_X \bar{X}$, then we have a commutative diagram:

$$\begin{array}{ccc} H^{2n}_Y(X, \mathbf{Z}/2\mathbf{Z}) & \to & H^{2n}_{\text{et}}(X, \mathbf{Z}/2\mathbf{Z}) \\ \downarrow \pi^* & & \downarrow \pi^* \\ H^{2n}_{\bar{Y}}(\bar{X}, \mathbf{Z}/2\mathbf{Z}) & \xrightarrow{r} & H^{2n}_{\text{et}}(\bar{X}, \mathbf{Z}/2\mathbf{Z}) \end{array}$$

where r is an isomorphism because $\bar{Y}$ consists of a single point (namely x, now regarded as an element of $X(\mathbf{C})$), and $r(\text{cl}(\bar{Y})) \neq 0$. Since $\pi^*\text{cl}(Y) = \text{cl}(\bar{Y})$ by [**6**, Cycle 2.3.8] (π: $\bar{X} \to X$ is the projection map), the map $H^{2n}_{\text{et}}(X, \mathbf{Z}/2\mathbf{Z}) \to H^{2n}_{\text{et}}(\bar{X}, \mathbf{Z}/2\mathbf{Z})$ is nonzero. $\square$

3. The local topological degree of a mapping germ is easily determined using cohomology. Let f: $(B_\delta, 0) \to (B_\varepsilon, 0)$ be a continuous map, where B_δ and B_ε are closed balls about 0 in $\mathbf{R}^n$, and $f^{-1}(0) = 0$. Then:

$$f^*: H^{n-1}(B_\varepsilon - \{0\}, \mathbf{Z}) \to H^{n-1}(B_\delta - \{0\}, \mathbf{Z}) \tag{3}$$

is multiplication by $\deg_0 f$, the local topological degree of f.

We will show that the algebraic version of this procedure (which uses étale cohomology) does not determine $\deg_0 f$ for $n > 1$. The algebraic formulation uses the henselization X of $A^n_{\mathbf{R}}$ at 0 (X is a canonical "smallest nbd" of 0 in $A^n_{\mathbf{R}}$). A finite mapping germ means a finite map f: $X \to X$ with $f(0) = 0$.

First, let us show that such an f does have a well-defined local topological degree. The descent theory of [7, §8] allows us to find a filtering category J of finite maps:

$$f_\alpha: (U_\alpha, 0) \to (V_\alpha, 0), \qquad f_\alpha^{-1}(0) = 0, \quad \alpha \in J,$$

between pointed étale nbds of 0 in $A^n_{\mathbf{R}}$, such that $X \simeq \varprojlim_J U_\alpha \simeq \varprojlim_J V_\alpha$ and $f = \varprojlim f_\alpha$. Each f_α induces a map f°_α: $(U_\alpha(\mathbf{R}), 0) \to (V_\alpha(\mathbf{R}), 0)$. Since the maps $U_\alpha(\mathbf{R}) \to \mathbf{R}^n$, $V_\alpha(\mathbf{R}) \to \mathbf{R}^n$ are local homeomorphisms, each f°_α has a local topological degree, and they are all the same because the f_α are compatible. We set $\deg_0 f = \deg_0 f^\circ_\alpha$, $\alpha \in J$.

The obvious analogy of (3) is to see if $\deg_0 f$ can be determined from the map:

$$f^*: H^*_{\text{et}}(X - \{0\}, M) \to H^*_{\text{et}}(X - \{0\}, M) \tag{4}$$

where M is some locally constant sheaf on X with finite fibers. This procedure works over $\mathbf{C}$. If we let $\bar{X} = X \times_{\mathbf{R}} \mathbf{C}$, then f induces a map $\bar{f}$: $\bar{X} \to \bar{X}$. One easily shows that $H^{2n-1}_{\text{et}}(\bar{X} - \{0\}, \mathbf{Z}/m\mathbf{Z}) \simeq \mathbf{Z}/m\mathbf{Z}$ and the methods below then show that $\bar{f}^*$, in dimension $2n - 1$, is multiplication by $\deg_0 \bar{f}$ (which is also the degree of f as a finite map). It is known that $\deg_0 f \equiv \deg_0 \bar{f} \bmod 2$.

The map (4), however, does *not* determine $\deg_0 f$:

Proposition 3.1. *If $n > 1$, the map f^* of (4) is determined uniquely by* $\deg_0 \bar{f}$.

Proof. X has the same étale homotopy type as Spec($\mathbf{R}$), so M is given by an action of $G = \mathbf{Z}/2\mathbf{Z}$ on some finite group (we often omit M in writing cohomology).

For each $\alpha \in J$, we can pick closed balls $B_\alpha \subseteq U_\alpha(\mathbf{C})$ and $B'_\alpha \subseteq V_\alpha(\mathbf{C})$ about 0 which are invariant under G, compatible with J, and which satisfy $f_\alpha(B_\alpha) \subseteq B'_\alpha$. Then we get a commutative diagram:

$$\begin{tikzcd}
H^q_{\mathrm{et}}(X - \{0\}) & \varinjlim_{J^\circ} H^q_{\mathrm{et}}(U_\alpha - \{0\}) \arrow[l, "\sim"'] \arrow[d, "\wr"] & H^q_{\mathrm{et}}(A^n_{\mathbf{R}} - \{0\}) \arrow[l, "\alpha"'] \arrow[dd, equal] \\
 & \varinjlim_{J^\circ} H^q((U_\alpha(\mathbf{C}) - \{0\})_G) \arrow[d, "\beta"] & \\
 & \varinjlim_{J^\circ} H^q((B_\alpha - \{0\})_G) & H^q((\mathbf{C}^n - \{0\})_G) \arrow[l, "\sim"'] \\
 & H^q((B_\alpha - \{0\})_G) \arrow[u, "\wr"] &
\end{tikzcd}$$

Since $H^q_{\mathrm{et}}(A^n_{\mathbf{R}} - \{0\}) \to H^q_{\mathrm{et}}(X - \{0\})$ is an isomorphism (use [**1**, XV 2.2] and the L.E.S. for local cohomology), the maps α and hence β are isomorphisms. We get a similar diagram for the B'_α's and then f and the f_α's give us a commutative diagram:

$$\begin{tikzcd}
H^q_{\mathrm{et}}(X - \{0\}) \arrow[r, "f^*"] \arrow[d, "\wr"] & H^q_{\mathrm{et}}(X - \{0\}) \arrow[d, "\wr"] \\
H^q((B'_\alpha - \{0\})_G) \arrow[r, "f^*_\alpha"] & H^q((B_\alpha - \{0\})_G)
\end{tikzcd}$$

Thus, we are reduced to a topological problem. Let $Y = S^{2n-1} \subseteq \mathbf{C}^n$ be the unit sphere with the G-action given by conjugation, and let $f\colon Y \to Y$ be an equivariant map. Let d be the degree of f (as a map of $2n - 1$-spheres). f induces a map $f_G\colon Y_G \to Y_G$, and we will show that $f^*_G\colon H^q(Y_G) \to H^q(Y_G)$ is determined by d if $n > 1$.

Y_G is the middle term of a fibration:

$$Y \to Y_G \xrightarrow{p} B_G$$

and the Serre spectral sequence gives us a long exact sequence:

$$\to H^q(G, M) \xrightarrow{p^*} H^q(Y_G, M) \to H^{q-2n+1}(G, H^{2n-1}(Y, M)) \to . \qquad (5)$$

Any point $x \in Y^G = S^{n-1}$ gives a section $s_x\colon B_G \to Y_G$ of p. Thus (5) becomes a split short exact sequence, and we have an isomorphism:

$$H^q(Y_G, M) \xrightarrow{\sim} H^q(G, M) \oplus H^{q-2n+1}(G, H^{2n-1}(Y, M))$$

which is canonical because the map to the first factor, $s^*_x\colon H^q(Y_G) \to H^q(B_G)$, does not depend on which $x \in Y^G = S^{n-1}$ we choose: since $n > 1$, S^{n-1} is connected. In particular, f^*_G respects this decomposition. On the first factor, f^*_G

induces the identity because f_G commutes with p. On the second factor, f_G^* is determined by f^*, so that there f_G^* is multiplication by d. $\square$

When $n = 1$, the situation is quite different. Here $f\colon X \to X$ can be written $f(x) = \pm\, x^k u^2$, where u is a unit. Then $\deg_0 \bar{f} = k$ and:

$$\deg_0 f = \begin{cases} 0, & \text{if } k \text{ is even,} \\ +1, & k \text{ odd}, f = x^k u^2, \\ -1, & k \text{ odd}, f = -x^k u^2. \end{cases}$$

Furthermore, $\deg_0 f$ is easy to detect from (4): for $q \geqslant 1$,

$$H^q_{\mathrm{et}}(X - \{0\}, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\sim} H^q((X(\mathbf{R}) - \{0\}) \times B_G, \mathbf{Z}/2\mathbf{Z}) \simeq (\mathbf{Z}/2\mathbf{Z})^2,$$

and one easily shows:

$\deg_0 f = 0 \Leftrightarrow f^*$ is not an isomorphism.

$\deg_0 f = 1 \Leftrightarrow f^*$ is the identity.

$\deg_0 f = -1 \Leftrightarrow f^*$ is an isomorphism $\neq$ the identity.

In a future paper we will investigate other ways étale cohomology might be used to determine $\deg_0 f$.

REFERENCES

1. M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lecture Notes in Math., vol. 305, Springer-Verlag, Berlin and New York, 1973.

2. M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Math., vol. 100, Springer-Verlag, Berlin and New York, 1970.

3. M. Artin and J. L. Verdier, *Seminar on étale cohomology of number fields*, Mimeographed notes, Woods Hole, 1964.

4. G. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

5. D. Cox, *Homotopy theory of simplicial schemes*, Compositio Math. (to appear).

6. P. Deligne et al., *Cohomologie étale*, Lecture Notes in Math., vol. 569, Springer-Verlag, Berlin and New York, 1977.

7. J. Dieudonné and A. Grothendieck, *Eléments de géométrie algébrique*. IV, Inst. Hautes Études Sci. Publ. Math. No. 28 (1966).

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903