MODULAR GROUP ALGEBRAS OF TOTALLY PROJECTIVE $p$-PRIMARY GROUPS

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ABSTRACT. Let $F$ be a field of characteristic $p > 0$ and let $G$ be a totally projective abelian $p$-group of countable $p$-length. If $FG \cong FH$ for some group $H$, then it is shown that $G \cong H$.

Let $F$ be a field of characteristic $p > 0$ and let $G$ be an (infinite) abelian $p$-group. The question of whether $FG \cong FH$ implies that $G \cong H$ has been considered by Berman and Mollov [1] and May [5]. (For the finite non-abelian case, see Passman [8].) It is shown in [1], [5] and by Dubois and Sehgal [2] that the Ulm-Kaplansky invariants of $G$ and $H$ must be equal. Thus one can easily settle the case when $G$ is countable. Since the totally projective $p$-groups form the largest natural class of abelian $p$-groups that are determined by their Ulm-Kaplansky invariants, one would like to know in general whether $H$ must be totally projective if $G$ is. In this direction, Berman and Mollov give an affirmative answer if $G$ is a direct sum of cyclic groups. By utilizing a theorem of Hill [4], we shall extend these results to $p$-groups of countable $p$-length. Precisely, we shall prove:

THEOREM 1. Let $R$ be a commutative ring with identity in which $p$ is not invertible, and let $\Omega$ be the first uncountable ordinal. Let $G$ be an abelian $p$-group of $p$-length $< \Omega$ whose reduced part is totally projective. If $H$ is a group such that $RG$ and $RH$ are isomorphic algebras, then $H$ is isomorphic to $G$.

To deduce Theorem 1, we shall prove several facts about the group of units of $RG$. For any commutative ring $R$ with identity and group $G$, let $U(RG)$ denote the group of units of $RG$ that have augmentation 1 (i.e., the coefficients sum to 1). We prove:

THEOREM 2. Let $R$ be a commutative ring with identity of characteristic $p$ that is perfect (i.e., every element is a $p$th power). Let $G$ be an abelian $p$-group of $p$-length $< \Omega$ whose reduced part is totally projective. Then $U(RG)$ is an abelian $p$-group of the same $p$-length as $G$, and whose reduced part is totally projective. Moreover, $G$ is a direct factor of $U(RG)$.

We remark that Mollov [6], [7] has investigated the structure of unit groups under certain restrictions.
Before proving two preliminary lemmas, we must briefly discuss $p$-heights. Groups will be written multiplicatively, thus we shall use $G^p$ ($\sigma$ an ordinal) to denote the subgroup of $G$ consisting of elements with $p$-heights $\geq \sigma$. Suppose that $R$ is a commutative ring with identity of characteristic $p$, and let $G$ be an abelian $p$-group. Then $U(RG)$ is a $p$-group and consists precisely of the elements of $RG$ of augmentation 1. Suppose further now that $R$ is perfect. Then $U(RG)^p = U(R(G^p))$, thus it follows that $U(RG)^\sigma = U(R(G^\sigma))$ for every ordinal $\sigma$. Let $\alpha \in U(RG)$ and consider the support of $\alpha$, i.e. those $g \in G$ that appear nontrivially in $\alpha$. Clearly the $p$-height of $\alpha$ in $U(RG)$ is the minimum of the $p$-heights in $G$ of the elements in the support of $\alpha$. In particular, if we regard $G$ as a subgroup of $U(RG)$, then it is an isotype subgroup. We also observe that the $p$-length of $U(RG)$ is clearly equal to the $p$-length of $G$.

**Lemma 1.** Let $R$ be a perfect commutative ring with identity of characteristic $p$, and let $G$ be an abelian $p$-group.

(a) Let $A$ be a finite subgroup of $G$, and let $N$ be a subgroup of $U(RA)$. Then $N$ is a nice subgroup of $U(RG)$.

(b) If $G$ is countable, then the reduced part of $U(RG)$ is totally projective.

**Proof.** Let $\alpha \in U(RG)$ and let $S = \{ ga | g \in (\text{support } \alpha), a \in A \}$. If $\beta \in N$, then the $p$-height of $\alpha\beta$ equals the $p$-height of some element in the finite set $S$. Therefore $\beta$ can be chosen to maximize this $p$-height. Hence $N$ is nice in $U(RG)$.

Now suppose that $G$ is countable, and let $A_1 \subseteq A_2 \subseteq \cdots$ be finite subgroups of $G$ with $\bigcup_{i<\omega} A_i = G$. For each $i$, we can choose a family $(N_{i\alpha})$ of subgroups of $U(RG)$ indexed by an initial segment of ordinals such that $U(RA_i) \subseteq N_{i\beta} \subseteq N_{i\alpha} \subseteq U(RA_{i+1})$ if $\beta < \alpha$, $N_{i\alpha} = \bigcup_{\beta < \alpha} N_{i\beta}$ if $\alpha$ is a limit ordinal,

$$N_{i0} = U(RA_i), \quad \bigcup_{\alpha} N_{i\alpha} = U(RA_{i+1}), \quad \text{and } (N_{i\alpha+1}; N_{i\alpha}) = p.$$  

Each $N_{i\alpha}$ is nice in $U(RG)$ by (a), therefore $\bigcup_{i<\omega}(N_{i\alpha})$ is a nice decomposition series for $U(RG)$. It follows that the reduced part of $U(RG)$ is totally projective.

**Lemma 2.** Let $R$ be a commutative ring with identity and let $G$ be an abelian group. Suppose that there exists a group $B$ such that $G$ is isomorphic to a direct factor of $U(RB)$. Then $G$ is a direct factor of $U(RG)$.

**Proof.** Let $G$ be isomorphic to the direct factor $V$ of $U(RB)$. Regarding $V$ as a subgroup of $U(RB)$, the isomorphism $G \rightarrow V$ induces a ring homomorphism $RG \rightarrow RB$, and hence a homomorphism $U(RG) \rightarrow U(RB)$. By composing with a projection of $U(RB)$ onto $V$, we obtain a homomorphism $U(RG) \rightarrow V$ that maps $G$ isomorphically to $V$. Thus $G$ is a direct factor of $U(RG)$.

**Proof of Theorem 2.** We have already observed that $U(RG)$ is an abelian
p-group of the same p-length as \(G\). We shall now show that the reduced part of \(U(RG)\) is totally projective by using induction on \(|G|\). The countable case is done by Lemma 1, therefore we may suppose that \(G\) is uncountable. The assumption that \(G\) has totally projective reduced part of p-length < \(\Omega\) means precisely that \(G\) is isomorphic to a direct sum of countable p-groups (see [3, Theorem 82.4]). Let \(\tau\) be the first ordinal with \(|\tau| = |G|\). We may assume that \(G = \Pi_{\beta < \tau} A_{\beta}\), where each \(A_{\beta}\) is countable. For each \(\alpha < \tau\), put \(G_{\alpha} = \Pi_{\beta < \alpha} A_{\beta}\) and \(U_{\alpha} = U(RG_{\alpha})\). (Note that \(G_0 = 1\).) The projection \(G_{\alpha+1} \to G_{\alpha}\) with kernel \(A_{\alpha}\) induces a surjective map \(U_{\alpha+1} \to U_{\alpha}\) that is idempotent. Thus we obtain an inner direct splitting \(U_{\alpha+1} = U_{\alpha} \times K_{\alpha}\). It is easily seen that \(U(RG) = \Pi_{\alpha < \tau} K_{\alpha}\). But \(|\alpha + 1| < |G|\), thus \(|U_{\alpha+1}| < |G|\), and consequently the reduced part of \(U_{\alpha+1}\) is totally projective by induction. It now follows that \(U(RG)\) has totally projective reduced part.

To show that \(G\) is a direct factor of \(U(RG)\) let \(B = \Pi_{i < \alpha} G_i\). Then the reduced part of \(U(RB)\) is totally projective by what we have just shown. Let \(f_\sigma(G)\) denote the \(\sigma\)th Ulm-Kaplansky invariant of \(G\). We know that \(f_\sigma(B) < f_\sigma(U(RB))\) since \(B\) is isotype in \(U(RB)\), therefore \(f_\sigma(U(RB)) = f_\sigma(G) + f_\sigma(U(RB))\) for every ordinal \(\sigma\). Moreover, if \(D_G\) and \(D_U\) are the maximal divisible subgroups of \(G\) and \(U(RB)\), respectively, then \(D_U \cong D_G \times D_U\). Hence, \(U(RB) \cong G \times U(RB)\) since we are dealing with totally projective reduced parts. By Lemma 2, we conclude that \(G\) is a direct factor of \(U(RG)\).

Since the complement of \(G\) in \(U(RG)\) has totally projective reduced part, one could determine it up to isomorphism by computing the rank of its maximal divisible subgroup and its Ulm-Kaplansky invariants in terms of \(R\) and the Ulm-Kaplansky invariants of \(G\). We remark that this is a feasible computation, but we shall not discuss it here.

**Proof of Theorem 1.** Our assumptions on \(R\) guarantee that there is a homomorphism (not necessarily surjective) \(R \to F\), where \(F\) is an algebraically closed field of characteristic \(p\). It follows that \(FG \cong FH\). One easily sees that \(H\) must be an abelian \(p\)-group since a torsion-free element in \(H\) would be transcendental over \(F\), while an element of order relatively prime to \(p\) would yield a nontrivial idempotent in \(FH\). Moreover, the maximal divisible subgroups of \(G\) and \(H\) are isomorphic by reference to [5, Corollary 7].

We may assume that the isomorphism \(FG \cong FH\) preserves augmentations, hence \(U(FG) \cong U(FH)\). Theorem 2 allows us to conclude that the reduced part of \(U(FH)\) is totally projective of countable \(p\)-length. Thus the reduced part of \(H\) is isomorphic to an isotype subgroup of countable \(p\)-length in the reduced part of \(U(FH)\). It now follows immediately from a result of Hill [4, Theorem 1] that the reduced part of \(H\) is totally projective. As was mentioned in the introduction, the Ulm-Kaplansky invariants of \(G\) and \(H\) are known to be equal; therefore \(G \cong H\).

We conclude with several questions and remarks. Does Theorem 1 hold if \(G\) has \(p\)-length \(\geq \Omega\)? In this case, we no longer have Hill’s theorem to apply.
Even more, there is uncertainty whether Theorem 2 holds if the $p$-length is $> \Omega$ since $G$ would not be a direct sum of countable groups. If one tries to prove Theorem 2 by induction on the $p$-length of $G$, the difficult step seems to occur at isolated ordinals. As a final question, if $G$ is a torsion-complete abelian $p$-group, then is $H$ torsion-complete? If so, then one could conclude $G$ and $H$ would be isomorphic since the Ulm-Kaplansky invariants serve to classify torsion-complete groups.

**REFERENCES**


