MODULAR GROUP ALGEBRAS OF TOTALLY PROJECTIVE 
p-PRIMARY GROUPS

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ABSTRACT. Let $F$ be a field of characteristic $p > 0$ and let $G$ be a totally projective abelian $p$-group of countable $p$-length. If $FG \cong FH$ for some group $H$, then it is shown that $G \cong H$.

Let $F$ be a field of characteristic $p > 0$ and let $G$ be an (infinite) abelian $p$-group. The question of whether $FG \cong FH$ implies that $G \cong H$ has been considered by Berman and Mollov [1] and May [5]. (For the finite non-abelian case, see Passman [8].) It is shown in [1], [5] and by Dubois and Sehgal [2] that the Ulm-Kaplansky invariants of $G$ and $H$ must be equal. Thus one can easily settle the case when $G$ is countable. Since the totally projective $p$-groups form the largest natural class of abelian $p$-groups that are determined by their Ulm-Kaplansky invariants, one would like to know in general whether $H$ must be totally projective if $G$ is. In this direction, Berman and Mollov give an affirmative answer if $G$ is a direct sum of cyclic groups. By utilizing a theorem of Hill [4], we shall extend these results to $p$-groups of countable $p$-length. Precisely, we shall prove:

**Theorem 1.** Let $R$ be a commutative ring with identity in which $p$ is not invertible, and let $\Omega$ be the first uncountable ordinal. Let $G$ be an abelian $p$-group of $p$-length $< \Omega$ whose reduced part is totally projective. If $H$ is a group such that $RG$ and $RH$ are isomorphic algebras, then $H$ is isomorphic to $G$.

To deduce Theorem 1, we shall prove several facts about the group of units of $RG$. For any commutative ring $R$ with identity and group $G$, let $U(RG)$ denote the group of units of $RG$ that have augmentation 1 (i.e., the coefficients sum to 1). We prove:

**Theorem 2.** Let $R$ be a commutative ring with identity of characteristic $p$ that is perfect (i.e., every element is a $p$th power). Let $G$ be an abelian $p$-group of $p$-length $< \Omega$ whose reduced part is totally projective. Then $U(RG)$ is an abelian $p$-group of the same $p$-length as $G$, and whose reduced part is totally projective. Moreover, $G$ is a direct factor of $U(RG)$.

We remark that Mollov [6], [7] has investigated the structure of unit groups under certain restrictions.
Before proving two preliminary lemmas, we must briefly discuss \( p \)-heights. Groups will be written multiplicatively, thus we shall use \( G^\alpha \) (\( \alpha \) an ordinal) to denote the subgroup of \( G \) consisting of elements with \( \alpha \)-heights \( \geq \alpha \). Suppose that \( R \) is a commutative ring with identity of characteristic \( p \), and let \( G \) be an abelian \( p \)-group. Then \( U(RG) \) is a \( p \)-group and consists precisely of the elements of \( RG \) of augmentation 1. Suppose further now that \( R \) is perfect. Then \( U(RG)^\alpha = U(R(G^\alpha)) \), thus it follows that \( U(RG)^\alpha = U(R(G^\alpha)) \) for every ordinal \( \alpha \). Let \( \alpha \in U(RG) \) and consider the support of \( \alpha \), i.e. those \( g \in G \) that appear nontrivially in \( \alpha \). Clearly the \( p \)-height of \( \alpha \) in \( U(RG) \) is the minimum of the \( p \)-heights in \( G \) of the elements in the support of \( \alpha \). In particular, if we regard \( G \) as a subgroup of \( U(RG) \), then it is an isotype subgroup. We also observe that the \( p \)-length of \( U(RG) \) is clearly equal to the \( p \)-length of \( G \).

**Lemma 1.** Let \( R \) be a perfect commutative ring with identity of characteristic \( p \), and let \( G \) be an abelian \( p \)-group.

(a) Let \( A \) be a finite subgroup of \( G \), and let \( N \) be a subgroup of \( U(RA) \). Then \( N \) is a nice subgroup of \( U(RG) \).

(b) If \( G \) is countable, then the reduced part of \( U(RG) \) is totally projective.

**Proof.** Let \( \alpha \in U(RG) \) and let \( S = \{ ga \mid g \in \text{(support } \alpha \text{)}, a \in A \} \). If \( \beta \in N \), then the \( p \)-height of \( \alpha \beta \) equals the \( p \)-height of some element in the finite set \( S \). Therefore \( \beta \) can be chosen to maximize this \( p \)-height. Hence \( N \) is nice in \( U(RG) \).

Now suppose that \( G \) is countable, and let \( A_1 \subseteq A_2 \subseteq \cdots \) be finite subgroups of \( G \) with \( \bigcup_{i<\omega} A_i = G \). For each \( i \), we can choose a family \( \{ N_{i\alpha} \} \) of subgroups of \( U(RG) \) indexed by an initial segment of ordinals such that \( U(RA_i) \subseteq N_{i\beta} \subseteq N_{i\alpha} \subseteq U(RA_{i+1}) \) if \( \beta < \alpha \), \( N_{i\alpha} = \bigcup_{\beta < \alpha} N_{i\beta} \) if \( \alpha \) is a limit ordinal,

\[
N_{i0} = U(RA_i), \quad \bigcup_\alpha N_{i\alpha} = U(RA_{i+1}), \quad \text{and} \quad (N_{i\alpha+1} : N_{i\alpha}) = p.
\]

Each \( N_{i\alpha} \) is nice in \( U(RG) \) by (a), therefore \( \bigcup_{i<\omega}(N_{i\alpha}) \) is a nice decomposition series for \( U(RG) \). It follows that the reduced part of \( U(RG) \) is totally projective.

**Lemma 2.** Let \( R \) be a commutative ring with identity and let \( G \) be an abelian group. Suppose that there exists a group \( B \) such that \( G \) is isomorphic to a direct factor of \( U(RB) \). Then \( G \) is a direct factor of \( U(RG) \).

**Proof.** Let \( G \) be isomorphic to the direct factor \( V \) of \( U(RB) \). Regarding \( V \) as a subgroup of \( U(RB) \), the isomorphism \( G \to V \) induces a ring homomorphism \( RG \to RB \), and hence a homomorphism \( U(RG) \to U(RB) \). By composing with a projection of \( U(RB) \) onto \( V \), we obtain a homomorphism \( U(RG) \to V \) that maps \( G \) isomorphically to \( V \). Thus \( G \) is a direct factor of \( U(RG) \).

**Proof of Theorem 2.** We have already observed that \( U(RG) \) is an abelian
p-group of the same p-length as G. We shall now show that the reduced part of \( U(RG) \) is totally projective by using induction on \(|G|\). The countable case is done by Lemma 1, therefore we may suppose that G is uncountable. The assumption that G has totally projective reduced part of p-length < \( \Omega \) means precisely that G is isomorphic to a direct sum of countable p-groups (see [3, Theorem 82.4]). Let \( \tau \) be the first ordinal with \(|\tau| = |G|\). We may assume that \( G = \Pi_{\beta<\tau} A_\beta \), where each \( A_\beta \) is countable. For each \( \alpha < \tau \), put \( G_\alpha = \Pi_{\beta<\alpha} A_\beta \) and \( U_\alpha = U(RG_\alpha) \). (Note that \( G_0 = 1 \).) The projection \( G_{\alpha+1} \to G_\alpha \) with kernel \( A_\alpha \) induces a surjective map \( U_{\alpha+1} \to U_\alpha \) that is idempotent. Thus we obtain an inner direct splitting \( U_{\alpha+1} = U_\alpha \times K_\alpha \). It is easily seen that \( U(RG) = \Pi_{\alpha<\tau} K_\alpha \). But \(|\alpha + 1| < |G|\), thus \(|U_{\alpha+1}| < |G|\), and consequently the reduced part of \( U_{\alpha+1} \) is totally projective by induction. It now follows that \( U(RG) \) has totally projective reduced part.

To show that G is a direct factor of \( U(RG) \) let \( B = \Pi_{i<\tau} G_i \). Then the reduced part of \( U(RB) \) is totally projective by what we have just shown. Let \( f_\sigma(G) \) denote the \( \sigma \)th Ulm-Kaplansky invariant of G. We know that \( f_\sigma(B) < f_\sigma(U(RB)) \) since B is isotype in \( U(RB) \), therefore \( f_\sigma(U(RB)) = f_\sigma(B) + f_\sigma(U(RB)) \) for every ordinal \( \sigma \). Moreover, if \( D_G \) and \( D_U \) are the maximal divisible subgroups of G and \( U(RB) \), respectively, then \( D_U \cong D_G \times D_U \). Hence, \( U(RB) \cong G \times U(RB) \) since we are dealing with totally projective reduced parts. By Lemma 2, we conclude that G is a direct factor of \( U(RG) \).

Since the complement of G in \( U(RG) \) has totally projective reduced part, one could determine it up to isomorphism by computing the rank of its maximal divisible subgroup and its Ulm-Kaplansky invariants in terms of R and the Ulm-Kaplansky invariants of G. We remark that this is a feasible computation, but we shall not discuss it here.

**Proof of Theorem 1.** Our assumptions on R guarantee that there is a homomorphism (not necessarily surjective) \( R \to F \), where F is an algebraically closed field of characteristic p. It follows that \( FG \cong FH \). One easily sees that H must be an abelian p-group since a torsion-free element in H would be transcendental over F, while an element of order relatively prime to p would yield a nontrivial idempotent in \( FH \). Moreover, the maximal divisible subgroups of G and H are isomorphic by reference to [5, Corollary 7].

We may assume that the isomorphism \( FG \cong FH \) preserves augmentations, hence \( U(FG) \cong U(FH) \). Theorem 2 allows us to conclude that the reduced part of \( U(FH) \) is totally projective of countable p-length. Thus the reduced part of H is isomorphic to an isotype subgroup of countable p-length in the reduced part of \( U(FH) \). It now follows immediately from a result of Hill [4, Theorem 1] that the reduced part of H is totally projective. As was mentioned in the introduction, the Ulm-Kaplansky invariants of G and H are known to be equal; therefore \( G \cong H \).

We conclude with several questions and remarks. Does Theorem 1 hold if G has p-length \( \geq \Omega \)? In this case, we no longer have Hill's theorem to apply.
Even more, there is uncertainty whether Theorem 2 holds if the $p$-length is $> \Omega$ since $G$ would not be a direct sum of countable groups. If one tries to prove Theorem 2 by induction on the $p$-length of $G$, the difficult step seems to occur at isolated ordinals. As a final question, if $G$ is a torsion-complete abelian $p$-group, then is $H$ torsion-complete? If so, then one could conclude $G$ and $H$ would be isomorphic since the Ulm-Kaplansky invariants serve to classify torsion-complete groups.

REFERENCES


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