PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 76, Number 1, August 1979

COMPACT OPERATORS WITH ROOT VECTORS THAT SPAN

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ABSTRACT. A concrete example is given of a bounded, linear, compact, quasiaffinity T acting on a separable, infinite dimensional, Hilbert space $\mathcal{H}$ with the property that the eigenvectors of T span $\mathcal{H}$ but the root vectors of T^* span a subspace of $\mathcal{H}$ with infinite dimensional orthocomplement.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. It is well known, of course, that if T is a *compact* operator in $\mathcal{L}(\mathcal{H})$, then the spectrum $\sigma(T)$ of T is a countable set whose only possible accumulation point is zero, and every nonzero number in $\sigma(T)$ is an eigenvalue of T. If λ is any eigenvalue of T (zero or not), then the associated eigenspace $\mathcal{E}_\lambda(T)$ is a hyperinvariant subspace for T that is finite dimensional unless $\lambda = 0$. The *root space* $\mathcal{R}_\lambda(T)$ of T corresponding to λ, which consists of all vectors f in $\mathcal{H}$ such that $(T - \lambda)^n f = 0$ for some positive integer n, is another (in general, larger) hyperinvariant subspace for T, and if $\lambda \neq 0$ then $\mathcal{R}_\lambda(T)$ is also finite-dimensional. Thus the subspace of $\mathcal{H}$ spanned by the set of all eigenvectors of T is $\mathcal{E}(T) = \bigvee_{\lambda \in \sigma(T)} \mathcal{E}_\lambda(T)$, and the subspace spanned by the set of all root vectors of T is $\mathcal{R}(T) = \bigvee_{\lambda \in \sigma(T)} \mathcal{R}_\lambda(T)$. Furthermore, it is an easy consequence of the Fredholm theory that if $\lambda \neq 0$ is an eigenvalue of T, then $\bar{\lambda}$ is an eigenvalue of T^* and $\dim \mathcal{E}_\lambda(T) = \dim \mathcal{E}_{\bar{\lambda}}(T^*)$, $\dim \mathcal{R}_\lambda(T) = \dim \mathcal{R}_{\bar{\lambda}}(T^*)$, so that if T has a good supply of eigenvectors [root vectors] corresponding to nonzero eigenvalues, then the same is true of T^*. (The compactness of T is essential here, of course.) This leads to the following questions. If T is compact and the eigenvectors [root vectors] of T span $\mathcal{H}$, must the same be true for T^*? It is known [**2**] and not difficult to see (cf. Proposition 1.2 below) that the answer to both questions is "no", roughly because T may have a large kernel and T^* only trivial kernel. But suppose that we impose the additional hypothesis that T be a quasiaffinity (i.e., that both T and T^* have trivial kernels). In this case the above questions are more difficult, and, moreover, it is known that if T is a compact quasiaffinity *all* of whose invariant subspaces are spanned by root vectors, then the same is true of T^* [**3**]. Nevertheless, as observed by Marcus [**3**], one can put the example of Hamburger [**2**] together with a result of Nikol'skii [**4**] to the effect that every compact operator is a part of a second compact operator whose root

Received by the editors June 20, 1978.
AMS (MOS) subject classifications (1970). Primary 47B05.
Key words and phrases. Compact, quasiaffinity, root vector.

vectors span its Hilbert space to conclude that the answer to both questions is still "no". But this situation is not completely satisfactory, because both the construction of Hamburger [2] and that of Nikol'skii [4] are rather complicated and what results is not so much a concrete example as an existence theorem. It is the purpose of this note to clarify this situation. We construct an explicit, fairly straightforward example of a quasiaffinity T in the Hilbert-Schmidt class with the property that $\mathfrak{S}(T) = \mathfrak{R}(T) = \mathcal{H}$ while $\dim(\mathcal{H} \ominus \mathfrak{R}(T^*)) = \aleph_0$.

LEMMA 1.1. *Suppose $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, and a mapping $T\colon \{e_n\} \to \mathcal{H}$ is defined in such a way that $\sum_{n=1}^{\infty} \|Te_n\|^2 < +\infty$. Then T has a unique extension to an operator $\tilde{T}$ in $\mathcal{L}(\mathcal{H})$, and $\tilde{T}$ belongs to the Hilbert-Schmidt class.*

PROOF. It suffices to show that T possesses an extension $\tilde{T}$ in $\mathcal{L}(\mathcal{H})$, and this goes as follows. By linearity, one may extend T to be defined at all finite linear combinations of the basis vectors. If f is any such vector, then a calculation using the Schwarz inequality shows that

$$\|Tf\| \leqslant \|f\| \left(\sum_{n=1}^{\infty} \|Te_n\|^2 \right)^{1/2},$$

and thus T may be extended to all of $\mathcal{H}$ by continuity.

We motivate the construction of our principal example by first giving a simpler example in which T is not a quasiaffinity.

PROPOSITION 1.2. *There exists a Hilbert-Schmidt operator T in $\mathcal{L}(\mathcal{H})$ such that $\mathfrak{S}(T) = \mathfrak{R}(T) = \mathcal{H}$ and such that $\mathfrak{R}(T^*) \neq \mathcal{H}$.*

PROOF. Let $\{g, f_1, f_2, f_3, \ldots\}$ be an orthonormal basis for $\mathcal{H}$ and define

$$Tg = 0,$$

$$Tf_n = \frac{1}{n} g + \frac{1}{n^2} f_n, \qquad n = 1, 2, \ldots.$$

It follows immediately from Lemma 1.1 that T can be extended to a Hilbert-Schmidt operator in $\mathcal{L}(\mathcal{H})$, which we continue to call T. If $h_n = g + f_n/n$, then

$$Th_n = \frac{1}{n^2}\left(g + \frac{1}{n} f_n\right) = \frac{1}{n^2} h_n, \qquad n = 1, 2, \ldots,$$

from which it follows that $h_n \in \mathfrak{S}(T)$ for all n. Since $\mathfrak{S}(T)$ is closed and $\|h_n - g\| \to 0$, $g \in \mathfrak{S}(T)$, and it follows easily that $f_n \in \mathfrak{S}(T)$ for all n. Thus $\mathfrak{S}(T) = \mathcal{H}$, and since $\mathfrak{R}(T) \supset \mathfrak{S}(T)$ by definition, we also have $\mathfrak{R}(T) = \mathcal{H}$. To show that $\mathfrak{R}(T^*) \neq \mathcal{H}$, we will show that g is orthogonal to $\mathfrak{R}(T^*)$. Note first that each h_n belongs to the range of T, and thus the range of T is dense in $\mathcal{H}$. Thus T^* has trivial kernel, and to complete the proof it suffices to show that if μ is a nonzero eigenvalue of T^*, n is a positive integer, and k is a vector such that

$$(T^* - \mu)^n k = 0, \tag{1}$$

then $(k, g) = 0$. It follows from (1) that there exists a polynomial p with $p(0) = 0$ such that $\mu^n k = p(T^*)k$. If we write p^* for the polynomial obtained from p by conjugating all its coefficients, then

$$\begin{aligned}(k, g) &= \frac{1}{\mu^n}(\mu^n k, g) = \frac{1}{\mu^n}(p(T^*)k, g)\\ &= \frac{1}{\mu^n}(k, p^*(T)g) = 0\end{aligned}$$

since g belongs to the kernel of T, and the proof is complete.

We turn now to the example that is the principal business of the note.

THEOREM 1.3. *There exists a quasiaffinity T in $\mathcal{L}(\mathcal{H})$ that belongs to the Hilbert-Schmidt class and satisfies $\mathfrak{S}(T) = \mathfrak{R}(T) = \mathcal{H}$ and* $\dim(\mathcal{H} \ominus \mathfrak{R}(T^*)) = \aleph_0$.

PROOF. We consider an orthonormal basis of $\mathcal{H}$ indexed as follows:

$$\begin{array}{l} f_{3,1}, f_{3,2}, f_{3,3},\\ f_{4,1}, f_{4,2}, f_{4,3}, f_{4,4},\\ \vdots \qquad\quad \vdots\\ f_{n,1}, f_{n,2}, \ldots, f_{n,n},\\ \vdots \qquad\quad \vdots\\ g_1, g_2, g_3, \ldots, g_n, \ldots \end{array}$$

Let T be given by

$$\begin{aligned} Tg_j &= \frac{1}{(j+1)!}\, g_{j+1}, \qquad 1 \leqslant j < \infty,\\ Tf_{n,j} &= \frac{1}{(j+1)!}\,\frac{1}{(n-j)}\, f_{n,j+1}, \qquad 1 \leqslant j < n,\ 3 \leqslant n < \infty,\\ Tf_{n,n} &= \frac{1}{n+1}\, g_1 - \frac{1}{n+1}\, g_{n+1} + \frac{1}{n(n+1)}\, f_{n,1}, \qquad 3 \leqslant n < \infty. \end{aligned} \tag{2}$$

Then

$$\begin{aligned} \sum_{j=1}^{\infty} \|Tg_j\|^2 &\leqslant \sum_{j=1}^{\infty} \frac{1}{(j+1)!} \leqslant e,\\ \sum_{n=3}^{\infty} \|Tf_{n,1}\|^2 &\leqslant \frac{1}{4} \sum_{n=3}^{\infty} \frac{1}{(n-1)^2} \leqslant \frac{\pi^2}{24},\\ \sum_{n=3}^{\infty} \|Tf_{n,2}\|^2 &\leqslant \frac{1}{36} \sum_{n=3}^{\infty} \frac{1}{(n-2)^2} \leqslant \frac{\pi^2}{216}, \end{aligned}$$

and

$$\sum_{j=3}^{\infty} \sum_{n=j}^{\infty} \|Tf_{n,j}\|^2$$

$$\leqslant \sum_{j=3}^{\infty} \left[\frac{1}{j^2(j+1)^2} + \frac{2}{(j+1)^2} + \sum_{n=j+1}^{\infty} \frac{1}{[(j+1)!]^2(n-j)^2} \right]$$

$$\leqslant \frac{3\pi^2}{6} + \frac{\pi^2}{6} \sum_{j=3}^{\infty} \frac{1}{[(j+1)!]^2} \leqslant \frac{3\pi^2}{6} + \frac{e\pi^2}{6}.$$

Thus, by Lemma 1.1, there exists a unique Hilbert-Schmidt operator in $\mathcal{L}(\mathcal{H})$ (which we continue to call T) that extends T. We first show that T is a quasiaffinity, i.e., that the kernel of T is trivial and the range of T is dense in $\mathcal{H}$. For the first part, let

$$x = \sum_{j=1}^{\infty} \beta_j g_j + \sum_{n=3}^{\infty} \sum_{j=1}^{n} \gamma_{n,j} f_{n,j}$$

and suppose $Tx = 0$. Then, expanding and collecting terms using (2), we obtain

$$0 = \left(\sum_{k=3}^{\infty} \frac{\gamma_{k,k}}{k+1} \right) g_1 + \frac{\beta_1}{2} g_2 + \frac{\beta_2}{6} g_3 + \sum_{j=4}^{\infty} \left(\frac{\beta_{j-1}}{j!} - \frac{\gamma_{j-1,j-1}}{j} \right) g_j$$

$$+ \sum_{n=3}^{\infty} \left\{ \frac{\gamma_{n,n}}{n(n+1)} f_{n,1} + \sum_{j=2}^{n} \frac{\gamma_{n,j-1}}{j!\,(n-j+1)} f_{n,j} \right\}.$$

Since the coefficient of each basis vector must be zero, we obtain $\gamma_{n,n} = 0$, $3 \leqslant n < \infty$, and $\gamma_{n,j-1} = 0$, $3 \leqslant n < \infty$, $2 \leqslant j \leqslant n$, so all of the $\gamma_{n,j}$ must be zero. Furthermore $\beta_1 = \beta_2 = 0$, and looking at the coefficient of g_j, $4 \leqslant j < \infty$, we see that $\beta_3 = \beta_4 = \cdots = 0$. Thus $x = 0$ as desired. To see that the range of T is dense in $\mathcal{H}$, we define

$$h_{n,j} = \frac{1}{n(n-1)\cdots(n-j+1)} f_{n,j} + g_j, \qquad 1 \leqslant j \leqslant n, 3 \leqslant n < \infty,$$

and observe that it follows from (2) that

$$Th_{n,j} = \frac{1}{(j+1)!} h_{n,j+1}, \qquad 1 < j < n, 3 \leqslant n < \infty,$$

$$Th_{n,n} = \frac{1}{(n+1)!} h_{n,1}, \qquad 1 \leqslant n < \infty. \tag{3}$$

If we set $\mathcal{H}_n = \bigvee\{h_{n,1}, \ldots, h_{n,n}\}$, $3 \leqslant n < \infty$, then it is clear from (3) that $\mathcal{H}_n$ is an n-dimensional invariant subspace of T and that the operator $T|\mathcal{H}_n$ satisfies

$$(T|\mathcal{H}_n)^n = \frac{1}{2!3!\cdots(n+1)!} 1_{\mathcal{H}_n}.$$

Furthermore, it follows easily from the fact that $T|\mathcal{H}_n$ is a weighted rotation operator (see (3)) that no nonzero polynomial of degree less than n can annihilate $T|\mathcal{H}_n$. Thus $\lambda^n - 1/2!3! \cdots (n+1)!$ is the minimal polynomial as well as the characteristic polynomial of $T|\mathcal{H}_n$, and hence $T|\mathcal{H}_n$ has n distinct nonzero eigenvalues. This implies, in turn, that the eigenvectors of $T|\mathcal{H}_n$ span $\mathcal{H}_n$, and thus that $\mathcal{H}_n \subset \mathcal{E}(T)$. In particular, $h_{n,j} \in \mathcal{E}(T)$, $1 \leqslant j \leqslant n$, $3 \leqslant n < \infty$, and since $\lim_n \|h_{n,j} - g_j\| = 0$, $g_j \in \mathcal{E}(T)$, $1 \leqslant j < \infty$. Since $f_{n,j} = n(n-1) \cdots (n-j+1)(h_{n,j} - g_j)$, it follows that $f_{n,j} \in \mathcal{E}(T)$, $1 \leqslant j \leqslant n$, $3 \leqslant n < \infty$, and hence that $\mathcal{E}(T) = \mathcal{H}$. Since T has trivial kernel, it is clear that $\mathcal{E}(T) \subset (\text{range } T)^-$, and thus $(\text{range } T)^- = \mathcal{H}$ and T is a quasiaffinity. Moreover, since $\mathcal{R}(T) \supset \mathcal{E}(T)$, we have $\mathcal{E}(T) = \mathcal{R}(T) = \mathcal{H}$. To complete the proof, we show that $\mathcal{R}(T^*)$ is orthogonal to the subspace $\mathcal{G} = \bigvee\{g_1, g_2, g_3, \ldots\}$. It clearly suffices to show that if h is any root vector of T^* and k is any fixed positive integer, then $(h, g_k) = 0$. Thus, let λ be a (necessarily nonzero) eigenvalue of T^*, and let j be a positive integer such that $(T^* - \lambda)^j h = 0$. Expanding this equation, we see that there is a polynomial p such that $p(0) = 0$ and such that $\lambda^j h = p(T^*)h$. Factoring $p(\lambda)$ as $p(\lambda) = \lambda q(\lambda)$ and setting $\lambda^j = \mu$, we obtain $q(T^*)T^*h = \mu h$, and hence $T^{*n}q(T^*)^n h = \mu^n h$ for every positive integer n. Thus

$$\begin{aligned}
|(h, g_k)| &= \frac{1}{|\mu|^n}\, |(\mu^n h, g_k)| = \frac{1}{|\mu|^n}\, \left(T^{*n}q(T^*)^n h, g_k\right) \\
&= \frac{1}{|\mu|^n}\, \left|\left(h, q^*(T)^n T^n g_k\right)\right| \\
&\leqslant \left|\frac{1}{\mu}\right|^n \|h\|\, \|q^*(T)\|^n \frac{1}{(k+1)! \cdots (k+n)!} \\
&\leqslant \frac{|1/\mu|^n}{n!}\, \frac{\|q^*(T)\|^n}{(n+1)!}\, \|h\|
\end{aligned}$$

for every n satisfying $n > k$ by virtue of (3). Thus $(h, g_k) = 0$, and the proof is complete.

Corollary 1.4. *The operator T defined by* (2) *has no quasiaffine transform that is a normal operator in $\mathcal{L}(\mathcal{H})$. Thus there exist compact operators in $\mathcal{L}(\mathcal{H})$ whose eigenvectors span $\mathcal{H}$ that are not quasisimilar to a normal operator.*

Proof. Suppose T has a quasiaffine transform N that is a normal operator in $\mathcal{L}(\mathcal{H})$. Then, by definition, there exists an operator X in $\mathcal{L}(\mathcal{H})$ with trivial kernel and dense range such that $XT = NX$. Since the eigenvectors of T span $\mathcal{H}$, it follows from this equation that the eigenvectors of N also span $\mathcal{H}$, in other words, that N is diagonalizable. But then N^* is also diagonalizable, and by taking adjoints in this equation, we learn that the eigenvectors of T^* span $\mathcal{H}$, contrary to fact.

By means of a lengthy and tedious computation, which we do not wish to include, one can establish these additional facts about the above operator T.

PROPOSITION 1.5. *The operator T defined by* (2) *has the property that the only eigenvalues of T are the eigenvalues of the various $T|\mathcal{H}_n$. Moreover, if λ is any eigenvalue of T, then $\mathcal{E}_\lambda(T) = \mathcal{R}_\lambda(T)$ and this space is one-dimensional.*

Finally, we remark that Theorem 1.3 and Corollary 1.4 show that the proof of [**1**, Corollary 5.5] is not correct.

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