A SCHÖRDER-BERNSTEIN THEOREM IN BAER*-RINGS WITH LATTICE-THEORETIC PROOF

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ABSTRACT. The Schröder-Bernstein theorem (SB-theorem) for *-equivalence of projections of a Baer*-ring is known. Here, we will prove an SB-theorem for algebraic equivalence (Theorem A) as a consequence of a lattice-theoretic SB-theorem (Theorem B). Theorem A and the known result about *-equivalence will be derived from Theorem B.

An element $e$ of an involutive ring $A$ is called a projection if it is idempotent ($e^2 = e$) and selfadjoint ($e^* = e$). The relation $e = e f$ defines a partial ordering of projections, denoted $e \preceq f$. Projections $e, f$ of $A$ are said to be algebraically equivalent if there exist $x, y$ in $A$ such that $yx = e$ and $xy = f$, and *-equivalent if there exists $x$ in $A$ such that $x^* x = e$ and $x x^* = f$. We write $e \sim f$ for either of these relations, indicating in the context which of the relations is intended. The notation $e \lessdot f$ means that $e \sim f_0$ for some projection $f_0 \prec f$.

An involutive ring $A$ is called a Baer*-ring if, for every subset $S$ of $A$, the right annihilator of $S$ has the form $eA$, $e$ a projection [3].

**Theorem A.** Let $e, f$ be projections in a Baer*-ring $A$. If $e \preceq f$ and $f \preceq e$ for algebraic equivalence, then $e \sim f$.

The analogue of Theorem A for *-equivalence is known [4, Theorem]. We shall give a lattice-theoretic proof of Theorem A.

**Definition.** An orthocomplemented lattice $L$ is called orthomodular if its orthocomplementation $x \mapsto 9(x)$ ($x \in L$) satisfies the identity

$$b = a \lor (b \land 9(a))$$

for $a \prec b$ ($a, b \in L$).

**Lemma [5, Lemma 29.15].** Let $L$ be an orthomodular lattice with orthocomplementation $x \mapsto 9(x)$ ($x \in L$). For $a \in L$, $[0, a] = \{ x \in L : x \prec a \}$ is an orthomodular lattice for the orthocomplementation

$$x \mapsto 9_a(x) = a \land 9(x)$$

for $x \in [0, a]$. If $a \prec 9(b)$, we write $a \perp b$.

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Theorem B. Suppose the orthomodular lattice $L$ is complete and possesses an equivalence relation $\sim$ satisfying the following conditions: (1) if $a_1 \perp a_2$, $b_1 \perp b_2$ and $a_1 \sim b_1$ ($i = 1, 2$), then $a_1 \lor a_2 \sim b_1 \lor b_2$;
(2) If $a \preceq b$, then there exists an order-preserving mapping $\varphi$ of $[0, a]$ into $[0, b]$ such that $x \sim \varphi(x)$ for all $x \in [0, a]$. Then $a \preceq b$ and $b \preceq a$ imply $a \sim b$ for $a, b \in L$.

Proof. From condition (2), there exist order-preserving mappings $\varphi_1$ and $\varphi_2$ such that $\varphi_1$ mps $[0, a]$ into $[0, b]$ and $\varphi_2$ maps $[0, b]$ into $[0, a]$. Define a mapping $\varphi$ of $[0, b]$ into $[0, b]$ by the formula
$$\varphi = \theta_b \circ \varphi_1 \circ \theta_a \circ \varphi_2.$$
By the lemma, $\varphi$ is an order-preserving mapping in the complete lattice $[0, b]$; therefore, by [7, Theorem 1], there exists $c_0 \in [0, b]$ such that $\varphi(c_0) = c_0$. Since $\theta_b \circ \theta_a = 1$,
$$\theta_b(c_0) = \theta_b(\varphi(c_0)) = \varphi_1(\theta_a(\varphi_2(c_0))).$$
By condition (2), one has
$$\theta_b(c_0) = \varphi_1(\theta_a(\varphi_2(c_0))) \sim \theta_a(\varphi_2(c_0))$$
and $c_0 \sim \varphi_2(c_0)$. Therefore, by the condition (1),
$$b = c_0 \lor \theta_b(c_0) \sim \varphi_2(c_0) \lor \theta_a(\varphi_2(c_0)) = a.$$ Q.E.D.

Proof of Theorem A. The projections of $A$ form a complete orthomodular lattice with the orthocomplementation $e \mapsto 1 - e$. Algebraic equivalence satisfies condition (1) [3, Theorem 17], and condition (2) as a special case of [6, Lemma 2.2]. Quote Theorem B. Q.E.D.

Remark 1. $\ast$-equivalence satisfies condition (1) [3, Theorem 25] and condition (2) [3, Theorem 24]. Therefore the SB-theorem for $\ast$-equivalence [4, Theorem] holds by Theorem B.

Remark 2. By the order of set inclusion and orthocomplementation of usual complement, the family of all subsets of a set forms a complete orthomodular lattice. With slight modification, Theorem B gives a proof of the Bernstein theorem of set theory.

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References