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# A SCHRÖDER-BERNSTEIN THEOREM IN BAER*-RINGS WITH LATTICE-THEORETIC PROOF

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ABSTRACT. The Schröder-Bernstein theorem (SB-theorem) for *-equivalence of projections of a Baer*-ring is known.

Here, we will prove an SB-theorem for algebraic equivalence (Theorem A) as a consequence of a lattice-theoretic SB-theorem (Theorem B). Theorem A and the known result about *-equivalence will be derived from Theorem B.

An element $e$ of an involutive ring $A$ is called a *projection* if it is idempotent ($e^2 = e$) and selfadjoint ($e^* = e$). The relation $e = ef$ defines a partial ordering of projections, denoted $e \leqslant f$. Projections $e, f$ of $A$ are said to be *algebraically equivalent* if there exist $x,y$ in $A$ such that $yx = e$ and $xy = f$, and **-equivalent* if there exists $x$ in $A$ such that $x^*x = e$ and $xx^* = f$. We write $e \sim f$ for either of these relations, indicating in the context which of the relations is intended. The notation $e \lesssim f$ means that $e \sim f_0$ for some projection $f_0 \leqslant f$.

An involutive ring $A$ is called a *Baer*-ring* if, for every subset $S$ of $A$, the right annihilator of $S$ has the form $eA$, $e$ a projection [3].

THEOREM A. *Let $e, f$ be projections in a Baer*-ring $A$. If $e \lesssim f$ and $f \lesssim e$ for algebraic equivalence, then $e \sim f$.*

The analogue of Theorem A for *-equivalence is known [4, Theorem]. We shall give a lattice-theoretic proof of Theorem A.

DEFINITION. An orthocomplemented lattice $L$ is called *orthomodular* if its orthocomplementation $x \mapsto \theta(x)$ $(x \in L)$ satisfies the identity

$$b = a \vee (b \wedge \theta(a))$$

for $a \leqslant b$ $(a, b \in L)$.

LEMMA [5, Lemma 29.15]. *Let $L$ be an orthomodular lattice with orthocomplementation $x \mapsto \theta(x)$ $(x \in L)$. For $a \in L$, $[0, a] = \{x \in L: x \leqslant a\}$ is an orthomodular lattice for the orthocomplementation*

$$x \mapsto \theta_a(x) = a \wedge \theta(x)$$

*for $x \in [0, a]$. If $a \leqslant \theta(b)$, we write $a \perp b$.*

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THEOREM B. *Suppose the orthomodular lattice $L$ is complete and possesses an equivalence relation $\sim$ satisfying the following conditions*: (1) *if* $a_1 \perp a_2$, $b_1 \perp b_2$ *and* $a_i \sim b_i$ $(i = 1, 2)$, *then* $a_1 \vee a_2 \sim b_1 \vee b_2$;

(2) *If* $a \lesssim b$, *then there exists an order-preserving mapping* $\varphi$ *of* $[0, a]$ *into* $[0, b]$ *such that* $x \sim \varphi(x)$ *for all* $x \in [0, a]$.

*Then* $a \lesssim b$ *and* $b \lesssim a$ *imply* $a \sim b$ *for* $a, b \in L$.

PROOF. From condition (2), there exist order-preserving mappings $\varphi_1$ and $\varphi_2$ such that $\varphi_1$ msps $[0, a]$ into $[0, b]$ and $\varphi_2$ maps $[0, b]$ into $[0, a]$. Define a mapping $\varphi$ of $[0, b]$ into $[0, b]$ by the formula

$$\varphi = \theta_b \circ \varphi_1 \circ \theta_a \circ \varphi_2.$$

By the lemma, $\varphi$ is an order-preserving mapping in the complete lattice $[0, b]$; therefore, by [7, Theorem 1], there eists $c_0 \in [0, b]$ such that $\varphi(c_0) = c_0$. Since $\theta_b \circ \theta_b = 1$,

$$\theta_b(c_0) = \theta_b(\varphi(c_0)) = \varphi_1(\theta_a(\varphi_2(c_0))).$$

By condition (2), one has

$$\theta_b(c_0) = \varphi_1(\theta_a(\varphi_2(c_0))) \sim \theta_a(\varphi_2(c_0))$$

and $c_0 \sim \varphi_2(c_0)$. Therefore, by the condition (1),

$$b = c_0 \vee \theta_b(c_0) \sim \varphi_2(c_0) \vee \theta_a(\varphi_2(c_0)) = a. \quad \text{Q.E.D.}$$

PROOF OF THEOREM A. The projections of $A$ form a complete orthomodular lattice with the orthocomplementation $e \mapsto 1 - e$. Algebraic equivalence satisfies condition (1) [3, Theorem 17], and condition (2) as a special case of [6, Lemma 2.2]. Quote Theorem B. Q.E.D.

REMARK 1. *-equivalence satisfies condition (1) [3, Theorem 25] and condition (2) [3, Theorem 24]. Therefore the SB-theorem for *-equivalence [4, Theorem] holds by Theorem B.

REMARK 2. By the order of set inclusion and orthocomplementation of usual complement, the family of all subsets of a set forms a complete orthomodular lattice. With slight modification, Theorem B gives a proof of the Bernstein theorem of set theory.

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