DEFORMABILITY OF A SUBMANIFOLD IN A EUCLIDEAN SPACE WHOSE IMAGE BY THE GAUSS MAP IS FIXED

YOSIO MUTÔ

Abstract. In the present paper, deformations of an immersion of an $m$-dimensional manifold $M$ in Euclidean $n$-space $\mathbb{R}^n$ such that the Gauss image is fixed pointwise are studied. We call such deformations admissible deformations. The main theorem gives a necessary and sufficient condition for an immersed manifold to admit nontrivial admissible deformations.

1. Let $M$ be an $m$-dimensional $C^\infty$ manifold and $i: M \times I \to \mathbb{R}^n$ a $C^\infty$ immersion where $I$ is some interval, $0 \in I$. $i(t)$ is defined by $i(t)M = i(M \times t)$ and we write $i(t)M = M(t)$. $M(t)$ is called a deformation of $M(0)$. We assume the Gauss map $\Gamma: M(t) \to G(m, n - m)$ is regular for $t \in I$. Let $M(t)$ have the local expression

$$x^h = x^h(y^1, \ldots, y^m; t)$$

(1.1)

where $x^1, \ldots, x^n$ are the rectangular coordinates in $\mathbb{R}^n$ and $y^1, \ldots, y^m$ are local coordinates in $M$. We understand

$$\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad \frac{\partial}{\partial y^i} = \frac{\partial}{\partial y^j}, \quad \Phi = \frac{\partial \Phi}{\partial t},$$

$$h, i, j, \ldots = 1, \ldots, n; \quad \kappa, \lambda, \mu, \ldots = 1, \ldots, m.$$

Suppose there exists a $(1, 1)$-tensor field $a(y; t)$ satisfying

$$\partial_\lambda x^h = a^h\partial_\lambda x^h.$$  

(1.2)

Obviously the geometrical meaning of this equation is that the Gauss image of $M(t)$ is fixed against $t$ for each point of $M$. Let us call any such deformation $M(t)$ an admissible deformation in this paper. As usual let $g_{\mu\lambda}$ be the components of the Riemannian metric of $M(t)$ induced by the immersion, $\{a_{\mu}^\kappa\}$ be the Christoffel symbols, $\nabla$ be covariant differentiation with respect to the Christoffel symbols, $B_{\mu}^h = \partial x^h/\partial y^\mu$ and $H_{\mu\lambda}^h$ be the second fundamental tensor.

First we get from (1.2):

$$\partial_\mu \partial_\lambda x^h = \left\{ \begin{array}{c} \sigma \mu \lambda \end{array} \right\} \partial_\sigma x^h = \nabla_\mu a_\lambda^\sigma B_{\sigma}^h + a_\lambda^\sigma H_{\mu\sigma}^h.$$
Then in view of the identity \( \sum_a B^a \mathcal{H}_{\lambda a} = 0 \) we get
\[
\nabla_\lambda a^\lambda = \nabla_\lambda a^\mu,
\]
(1.3)
\[
a^\mu \mathcal{H}_{\lambda a}^h = a^\lambda \mathcal{H}_{\mu a}^h.
\]
(1.4)
Differentiating \( g_{\mu \lambda} = \sum_a \partial_\mu x^a \partial_\lambda x^a \) with respect to \( t \) we get
\[
h_{\mu \lambda} = a_{\mu \lambda} + a_{\lambda \mu}
\]
(1.5)
where \( a_{\mu \lambda} = a^\sigma g_{\sigma \epsilon \lambda} \). From this and
\[
\frac{\partial}{\partial t} \left( \partial_\nu g_{\sigma \mu \lambda} - \left\{ \frac{\kappa}{\nu} \right\} g_{\sigma \epsilon \lambda} - \left\{ \frac{\kappa}{\nu} \right\} g_{\epsilon \mu \lambda} \right) = 0
\]
we get
\[
\left( \nabla_\nu a^\nu - \left\{ \frac{\kappa}{\nu} \right\} \right) g_{\epsilon \lambda} + \left( \nabla_\nu a^\nu - \left\{ \frac{\kappa}{\nu} \right\} \right) g_{\epsilon \mu \lambda} = 0.
\]
This shows that the tensor defined by
\[
A_{\mu \lambda} = \left( \nabla_\nu a^\nu - \left\{ \frac{\kappa}{\nu} \right\} \right) g_{\epsilon \lambda}
\]
is skew symmetric in \( \mu \) and \( \lambda \). On the other hand, \( A_{\mu \lambda} \) is symmetric in \( \nu \) and \( \mu \) in view of (1.3). Hence we have
\[
\left\{ \frac{\kappa}{\mu \lambda} \right\} = \nabla_\mu a^\lambda.
\]
(1.6)
Differentiating
\[
H_{\mu \lambda}^h = \partial_\mu \partial_\lambda x^h - \left\{ \frac{\kappa}{\mu \lambda} \right\} \partial_\epsilon x^h
\]
and the Gauss equation
\[
K_{\mu \lambda \epsilon k} = \sum_h H_{\mu \epsilon k}^h H_{\lambda a}^h - \sum_h H_{\mu a}^h H_{\lambda k}^h
\]
with respect to \( t \) and taking (1.6) into account, we get
\[
\dot{H}_{\mu \lambda}^h = a^\sigma H_{\mu a}^h,
\]
(1.7)
\[
\dot{K}_{\mu \lambda \epsilon k} = a^\sigma K_{\mu \epsilon k} + a^\sigma K_{\mu \lambda \sigma}.
\]
(1.8)
2. From (1.8) we get:

**Theorem 2.1.** Assume in an admissible deformation \( M(t) \) that \( M(0) \) is a flat Riemannian manifold. Then \( M(t) \) is a flat Riemannian manifold.

From (1.5) and (1.6) we can easily prove:

**Theorem 2.2.** Assume in an admissible deformation \( M(t) \) that \( t \) induces an isometric deformation of \( M(t) \). Then the tensor field \( a^\lambda \) is covariantly constant and \( a_{\mu \lambda} = - a_{\lambda \mu} \).

If an admissible deformation \( M(t) \) is such that the tensor field \( a \) vanishes, then \( M(t) \) undergoes only parallel displacements. If \( (a^\lambda) \) is a scalar matrix, then \( M(t) \) changes homothetically. In such cases the admissible deformation is said to be trivial.
Let \( i_0 : M \to \mathbb{R}^n \) be a \( C^\infty \) immersion. If there exists a nontrivial admissible deformation \( M(t) \) such that \( M(0) = i_0 M \), then \( M(0) \) is said to be nontrivially deformable with the Gauss image fixed, or to admit a nontrivial admissible deformation.

A typical nontrivial deformation with fixed Gauss image is found in the case of \( n = m + 1 \), as is well known.

**Remark 1.** If \( n = m + 1 \), we have for any deformation \( \dot{x}^h = \xi N^h + t^\lambda B^h_\lambda \) where \( N \) is the unit normal vector field, \( \xi \) is a function on \( M \) and \( t^\lambda \) is a vector field on \( M \). Then we get

\[
\frac{\partial}{\partial t} \dot{x}^h = \left( \nabla_{\dot{x}^\xi} N^h + \xi \nabla_{\dot{x}^h} \right) + \left( \xi \dot{N}^h + \nabla_{\dot{x}^h} \right) B^h_\lambda.
\]

(1.2) is satisfied if we take \( t^\lambda \) such that \( \nabla_{\dot{x}^\xi} + t^\lambda \nabla_{\dot{x}^\mu} = 0 \) and put \( a^\lambda = \nabla_{\dot{x}^\mu} - \xi \dot{x}^\mu \).

A product immersion also admits a nontrivial admissible deformation.

**Remark 2.** By a product immersion we mean the case \( M = M_1 \times M_2 \), \( i(t)M = i_1(t)M_1 \times i_2(t)M_2 \). Then a nontrivial admissible deformation is obtained if we assume \( i_1(t)M_1 \) and \( i_2(t)M_2 \) are homothetic to \( i_1(0)M_1 \) and \( i_2(0)M_2 \), respectively.

### 3. We prove the following main theorem.

**Theorem 3.1.** Let \( M \) be a compact \( C^\infty \) manifold and let \( M_0 = i_0 M \) be a \( C^\infty \) immersion into \( \mathbb{R}^n \) expressed locally by \( x^h = \varphi^h(y^1, \ldots, y^m) \) where the \( x^h \) \((h = 1, \ldots, n)\) are rectangular coordinates and \( y^k \) \((k = 1, \ldots, m)\) are local coordinates of \( M \). Let \( b_\lambda^* \) be a \((1,1)\)-tensor field on \( M \) such that there exists no constant \( c \) satisfying \( b_\lambda^* = c \delta_\lambda^* \) and consider the following conditions:

(i) \( b_\lambda^\kappa \) satisfies the equations

\[
\nabla_\mu b_\lambda^\kappa = \nabla_\lambda b_\mu^\kappa, \quad b_\mu^\kappa H_{\lambda\sigma}^h = b_\lambda^\kappa H_{\mu\sigma}^h
\]

(3.1)

where the Riemannian connection is induced by the immersion \( i_0 \);

(ii) there exist functions \( V^h \) \((h = 1, \ldots, n)\) satisfying

\[
\partial_\mu V^h = b_\mu^\kappa \partial_\kappa \varphi^h.
\]

(3.2)

Then (a) condition (i) follows from (ii); (b) if \( M \) is simply connected, (i) is equivalent to (ii); (c) a necessary and sufficient condition that the immersion \( M_0 \) admits a nontrivial admissible deformation is that there exists a tensor field \( b_\lambda^* \) satisfying the condition (ii); (d) especially when \( M \) is simply connected, \( M_0 \) admits a nontrivial admissible deformation if there exists a tensor field \( b_\lambda^* \) satisfying (i).

**Proof.** (a) is obtained if we substitute (3.2) into \( \partial_\mu \partial_\lambda V^h - \partial_\lambda \partial_\mu V^h = 0 \). As the \( b_\mu^\kappa \partial_\kappa \varphi^h \) are closed forms on \( M \) if (3.1) holds, (b) follows immediately. (d) is obtained from (b) and (c). Hence we need only to prove (c) and especially the sufficiency, for (3.2) is a special case of (1.2).

For that purpose first consider the system of ordinary differential equations
with the initial condition \( a_\lambda^\kappa(0) = b_\lambda^\kappa \) at each point \( p \) of \( M \). Let \( A = (a_\lambda^\kappa) \) and \( B = (b_\lambda^\kappa) \). Then the solution is

\[
A = (E + Bt)^{-1}B
\]

where \( E \) is the unit matrix and the domain of \( t \) is a certain interval \((-\varepsilon_p, \varepsilon_p)\). \( M \) being compact, there exists a positive number \( \varepsilon \) such that \( \varepsilon < \varepsilon_p \) for all \( p \in M \). If we consider at each point of \( M \) the system of ordinary differential equations

\[
d\beta^h_\mu/dt = a_\mu^\sigma(t)\beta^h_\sigma,
\]

there exists also a unique solution \( \beta^h_\mu(t) \) satisfying the initial condition

\[
\beta^h_\mu(0) = \delta_\mu^h.
\]

From (3.3) and (3.4) we get \( d^2\beta^h_\mu/dt^2 = 0 \) and the solution of (3.4) is

\[
\beta^h_\mu = \beta^h_\mu(0) + a_\mu^\sigma(0)\beta^h_\sigma(0)t.
\]

Then in view of (3.2) and (3.5) we get

\[
\beta^h_\mu = \partial_\mu(\psi^h(y) + V^h(y)t)
\]

and the immersion given by

\[
x^h(y; t) = \psi^h(y) + V^h(y)t
\]

satisfies

\[
\partial^2x^h(y; t)/\partial y^\lambda\partial t = \partial_\lambda V^h.
\]

On the other hand we get, from (3.6) and (3.7), \( a_\mu^\sigma\partial_\sigma x^h(y; t) = a_\mu^\sigma\beta^h_\sigma \) which becomes on account of (3.4) and (3.6):

\[
a_\mu^\sigma\partial_\sigma x^h(y; t) = \partial_\mu V^h.
\]

From this result and (3.8) we get \( \partial^2x^h/\partial y^\lambda\partial t = a_\lambda^\kappa\partial_\kappa x^h \) and (c) is proved.

Q.E.D.

We want to give an application. The space of \((1, 1)\)-tensor fields \( b_\lambda^\kappa \) satisfying (3.1) is a vector space over \( R \) if the case \( b_\lambda^\kappa = c\delta_\kappa^\lambda \) is not excluded.

Let us denote it by \( b \). Let \( l = \dim b \) and \( h = \dim H^1(M) \). Then there exists a subspace \( b_1 \) of \( b \) of dimension at least \( l - h \) such that \( b_\lambda^\kappa \partial_\kappa x^1 \) is a gradient for each tensor field \( b_\lambda^\kappa \) of \( b_1 \). There exists also a subspace \( b_2 \) of \( b_1 \) of dimension at least \( l - 2h \) such that \( b_\lambda^\kappa \partial_\kappa x^2 \) is a gradient for each element \( b_\lambda^\kappa \) of \( b_2 \) and so on. Thus, if \( l - nh - 1 > 0 \) holds, there exists a tensor field \( b_\lambda^\kappa \) satisfying (ii) of Theorem 3.1. This proves:

Theorem 3.2. If \( l = \dim b \) and \( h = \dim H^1(M) \) satisfy \( l - nh - 1 > 0 \), then the immersion \( i_0 M \) admits a nontrivial admissible deformation.

4. Let \( M \) be a Veronese manifold immersed in a Euclidean space \( R^n \) in full. If \( H_\mu^h \) is the second fundamental tensor at a point \( p \) of \( M \), then for any vector \( u^h \) of \( R^n \) satisfying
\[\sum_{h} H_{\mu \lambda}^h u^h = 0,\]

we have \([1]\]

\[u^h = \tau^\lambda B^h_{\lambda}.\]

As we have \(n = m(m + 3)/2\) for the Veronese manifold, the above mentioned fact indicates that there exist among \(n\) symmetric \((m \times m)\) matrices \((H_{\mu \lambda}^h)\) just \(m(m + 1)/2\) linearly independent symmetric matrices. This proves that the matrix \((a^\mu_\lambda)\) satisfying (1.4) is a scalar matrix. Thus we have

**Theorem 4.1.** A Veronese manifold immersed in a Euclidean space in full admits no nontrivial admissible deformation.

This means that, when a Veronese manifold immersed in a Euclidean space in full undergoes a deformation other than a parallel displacement, a homothety, or their composition, then the Gauss image moves pointwise.

**5.** Let us consider an immersion \(i_0: M \to \mathbb{R}^n\) and an admissible deformation \(M(t), M(0) = i_0M, \) again. Let \(H_{\mu \lambda}^h\) be the second fundamental tensor of \(M(t)\) at \(i(t)p\) where \(p\) is fixed. When \(\mu\) and \(\lambda\) are given and \(h\) runs the range \(1, \ldots, n\) we get components of a vector of \(\mathbb{R}^n\). There are \(m^2\) such vectors and they span a vector space whose dimension we denote by \(d_p\). Then \(d_p\) satisfies \(d_p < m(m + 1)/2\) and \(d_p < n - m\). In view of (1.7) we get:

**Theorem 5.1.** \(d_p\) is an invariant of any admissible deformation.

**Bibliography**


2262-150, Tomioka-cho, Kanazawa-ku, Yokohama 236, Japan