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DEFORMABILITY OF A SUBMANIFOLD IN A EUCLIDEAN SPACE WHOSE IMAGE BY THE GAUSS MAP IS FIXED

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ABSTRACT. In the present paper, deformations of an immersion of an m-dimensional manifold M in Euclidean n-space R^n such that the Gauss image is fixed pointwise are studied. We call such deformations admissible deformations. The main theorem gives a necessary and sufficient condition for an immersed manifold to admit nontrivial admissible deformations.

1. Let M be an m-dimensional C^∞ manifold and i: $M \times I \to R^n$ a C^∞ immersion where I is some interval, $0 \in I$. $i(t)$ is defined by $i(t)M = i(M \times t)$ and we write $i(t)M = M(t)$. $M(t)$ is called a deformation of $M(0)$. We assume the Gauss map Γ: $M(t) \to G(m, n - m)$ is regular for $t \in I$. Let $M(t)$ have the local expression

$$x^h = x^h(y^1, \dots, y^m; t) \tag{1.1}$$

where $x^1, \dots, x^n$ are the rectangular coordinates in R^n and $y^1, \dots, y^m$ are local coordinates in M. We understand

$$\partial_i \Phi = \frac{\partial \Phi}{\partial x^i}, \qquad \partial_\lambda \Phi = \frac{\partial \Phi}{\partial y^\lambda}, \qquad \dot{\Phi} = \Phi^{\cdot} = \frac{\partial \Phi}{\partial t},$$

$$h, i, j, \dots = 1, \dots, n; \qquad \kappa, \lambda, \mu, \dots = 1, \dots, m.$$

Suppose there exists a (1, 1)-tensor field $a(y; t)$ satisfying

$$\partial_\lambda \dot{x}^h = a_\lambda{}^\sigma \partial_\sigma x^h. \tag{1.2}$$

Obviously the geometrical meaning of this equation is that *the Gauss image of $M(t)$ is fixed against t for each point of M*. Let us call any such deformation $M(t)$ an *admissible deformation* in this paper. As usual let $g_{\mu\lambda}$ be the components of the Riemannian metric of $M(t)$ induced by the immersion, $\{{}^{\kappa}_{\mu\lambda}\}$ be the Christoffel symbols, ∇ be covariant differentiation with respect to the Christoffel symbols, $B_\mu{}^h = \partial x^h / \partial y^\mu$ and $H_{\mu\lambda}{}^h$ be the second fundamental tensor.

First we get from (1.2):

$$\partial_\mu \partial_\lambda \dot{x}^h - \left\{ {\sigma \atop \mu\ \lambda} \right\} \partial_\sigma \dot{x}^h = \nabla_\mu a_\lambda{}^\sigma B_\sigma{}^h + a_\lambda{}^\sigma H_{\mu\sigma}{}^h.$$

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Then in view of the identity $\Sigma_h B_\nu{}^h H_{\mu\lambda}{}^h = 0$ we get

$$\nabla_\mu a_\lambda{}^\kappa = \nabla_\lambda a_\mu{}^\kappa, \tag{1.3}$$

$$a_\mu{}^\sigma H_{\lambda\sigma}{}^h = a_\lambda{}^\sigma H_{\mu\sigma}{}^h. \tag{1.4}$$

Differentiating $g_{\mu\lambda} = \Sigma_h \partial_\mu x^h \partial_\lambda x^h$ with respect to t we get

$$\dot{g}_{\mu\lambda} = a_{\mu\lambda} + a_{\lambda\mu} \tag{1.5}$$

where $a_{\mu\lambda} = a_\mu{}^\sigma g_{\sigma\lambda}$. From this and

$$\frac{\partial}{\partial t}\left(\partial_\nu g_{\mu\lambda} - \left\{ {\kappa \atop \nu\ \mu} \right\} g_{\kappa\lambda} - \left\{ {\kappa \atop \nu\ \lambda} \right\} g_{\kappa\mu}\right) = 0$$

we get

$$\left(\nabla_\nu a_\mu{}^\kappa - \left\{ {\kappa \atop \nu\ \mu} \right\}^{\cdot}\right) g_{\kappa\lambda} + \left(\nabla_\nu a_\lambda{}^\kappa - \left\{ {\kappa \atop \nu\ \lambda} \right\}^{\cdot}\right) g_{\kappa\mu} = 0.$$

This shows that the tensor defined by

$$A_{\nu\mu\lambda} = \left(\nabla_\nu a_\mu{}^\kappa - \left\{ {\kappa \atop \nu\ \mu} \right\}^{\cdot}\right) g_{\kappa\lambda}$$

is skew symmetric in μ and λ. On the other hand, $A_{\nu\mu\lambda}$ is symmetric in ν and μ in view of (1.3). Hence we have

$$\left\{ {\kappa \atop \mu\ \lambda} \right\}^{\cdot} = \nabla_\mu a_\lambda{}^\kappa. \tag{1.6}$$

Differentiating

$$H_{\mu\lambda}{}^h = \partial_\mu \partial_\lambda x^h - \left\{ {\kappa \atop \mu\ \lambda} \right\} \partial_\kappa x^h$$

and the Gauss equation

$$K_{\nu\mu\lambda\kappa} = \sum_h H_{\nu\kappa}{}^h H_{\mu\lambda}{}^h - \sum_h H_{\mu\kappa}{}^h H_{\nu\lambda}{}^h$$

with respect to t and taking (1.6) into account, we get

$$\dot{H}_{\mu\lambda}{}^h = a_\lambda{}^\sigma H_{\mu\sigma}{}^h, \tag{1.7}$$

$$\dot{K}_{\nu\mu\lambda\kappa} = a_\lambda{}^\sigma K_{\nu\mu\sigma\kappa} + a_\kappa{}^\sigma K_{\nu\mu\lambda\sigma}. \tag{1.8}$$

2. From (1.8) we get:

THEOREM 2.1. *Assume in an admissible deformation $M(t)$ that $M(0)$ is a flat Riemannian manifold. Then $M(t)$ is a flat Riemannian manifold.*

From (1.5) and (1.6) we can easily prove:

THEOREM 2.2. *Assume in an admissible deformation $M(t)$ that t induces an isometric deformation of $M(t)$. Then the tensor field $a_\mu{}^\lambda$ is covariantly constant and $a_{\mu\lambda} = - a_{\lambda\mu}$.*

If an admissible deformation $M(t)$ is such that the tensor field a vanishes, then $M(t)$ undergoes only parallel displacements. If $(a_\mu{}^\lambda)$ is a scalar matrix, then $M(t)$ changes homothetically. In such cases the admissible deformation is said to be *trivial*.

Let i_0: $M \to R^n$ be a C^∞ immersion. If there exists a nontrivial admissible deformation $M(t)$ such that $M(0) = i_0M$, then $M(0)$ is said to be nontrivially deformable with the Gauss image fixed, or to admit a nontrivial admissible deformation.

A typical nontrivial deformation with fixed Gauss image is found in the case of $n = m + 1$, as is well known.

REMARK 1. If $n = m + 1$, we have for any deformation $\dot{x}^h = \xi N^h + t^\lambda B_\lambda{}^h$ where N is the unit normal vector field, ξ is a function on M and t^λ is a vector field on M. Then we get

$$\partial_\mu \dot{x}^h = \left(\nabla_\mu \xi + t^\lambda h_{\mu\lambda}\right)N^h + \left(-\xi h_\mu{}^\lambda + \nabla_\mu t^\lambda\right)B_\lambda{}^h.$$

(1.2) is satisfied if we take t^λ such that $\nabla_\mu \xi + t^\lambda h_{\mu\lambda} = 0$ and put $a_\mu{}^\lambda = \nabla_\mu t^\lambda - \xi h_\mu{}^\lambda$.

A product immersion also admits a nontrivial admissible deformation.

REMARK 2. By a product immersion we mean the case $M = M_1 \times M_2$, $i(t)M = i_1(t)M_1 \times i_2(t)M_2$. Then a nontrivial admissible deformation is obtained if we assume $i_1(t)M_1$ and $i_2(t)M_2$ are homothetic to $i_1(0)M_1$ and $i_2(0)M_2$, respectively.

3. We prove the following main theorem.

THEOREM 3.1. *Let M be a compact C^∞ manifold and let $M_0 = i_0M$ be a C^∞ immersion into R^n expressed locally by $x^h = \varphi^h(y^1, \dots, y^m)$ where the x^h $(h = 1, \dots, n)$ are rectangular coordinates and y^κ $(\kappa = 1, \dots, m)$ are local coordinates of M. Let $b_\lambda{}^\kappa$ be a* (1, 1)*-tensor field on M such that there exists no constant c satisfying $b_\lambda{}^\kappa = c\delta_\lambda^\kappa$ and consider the following conditions*:

(i) *$b_\lambda{}^\kappa$ satisfies the equations*

$$\nabla_\mu b_\lambda{}^\kappa = \nabla_\lambda b_\mu{}^\kappa, \qquad b_\mu{}^\sigma H_{\lambda\sigma}{}^h = b_\lambda{}^\sigma H_{\mu\sigma}{}^h \tag{3.1}$$

where the Riemannian connection is induced by the immersion i_0;

(ii) *there exist functions V^h $(h = 1, \dots, n)$ satisfying*

$$\partial_\mu V^h = b_\mu{}^\sigma \partial_\sigma \varphi^h. \tag{3.2}$$

Then (a) *condition* (i) *follows from* (ii); (b) *if M is simply connected,* (i) *is equivalent to* (ii); (c) *a necessary and sufficient condition that the immersion M_0 admits a nontrivial admissible deformation is that there exists a tensor field $b_\lambda{}^\kappa$ satisfying the condition* (ii); (d) *especially when M is simply connected, M_0 admits a nontrivial admissible deformation if there exists a tensor field $b_\lambda{}^\kappa$ satisfying* (i).

PROOF. (a) is obtained if we substitute (3.2) into $\partial_\mu\partial_\lambda V^h - \partial_\lambda\partial_\mu V^h = 0$. As the $b_\mu{}^\sigma\partial_\sigma\varphi^h$ are closed forms on M if (3.1) holds, (b) follows immediately. (d) is obtained from (b) and (c). Hence we need only to prove (c) and especially the sufficiency, for (3.2) is a special case of (1.2).

For that purpose first consider the system of ordinary differential equations

$$da_\lambda{}^\kappa/dt + a_\lambda{}^\rho a_\rho{}^\kappa = 0 \tag{3.3}$$

with the initial condition $a_\lambda{}^\kappa(0) = b_\lambda{}^\kappa$ at each point p of M. Let $A = (a_\lambda{}^\kappa)$ and $B = (b_\lambda{}^\kappa)$. Then the solution is

$$A = (E + Bt)^{-1}B$$

where E is the unit matrix and the domain of t is a certain interval $(-\varepsilon_p, \varepsilon_p)$. M being compact, there exists a positive number ε such that $\varepsilon < \varepsilon_p$ for all $p \in M$. If we consider at each point of M the system of ordinary differential equations

$$d\beta_\mu{}^h/dt = a_\mu{}^\sigma(t)\beta_\sigma{}^h, \tag{3.4}$$

there exists also a unique solution $\beta_\mu{}^h(t)$ satisfying the initial condition

$$\beta_\mu{}^h(0) = \partial_\mu\varphi^h. \tag{3.5}$$

From (3.3) and (3.4) we get $d^2\beta_\mu{}^h/dt^2 = 0$ and the solution of (3.4) is

$$\beta_\mu{}^h = \beta_\mu{}^h(0) + a_\mu{}^\sigma(0)\beta_\sigma{}^h(0)t.$$

Then in view of (3.2) and (3.5) we get

$$\beta_\mu{}^h = \partial_\mu(\varphi^h(y) + V^h(y)t) \tag{3.6}$$

and the immersion given by

$$x^h(y; t) = \varphi^h(y) + V^h(y)t \tag{3.7}$$

satisfies

$$\partial^2 x^h(y; t)/\partial y^\lambda \partial t = \partial_\lambda V^h. \tag{3.8}$$

On the other hand we get, from (3.6) and (3.7), $a_\mu{}^\sigma\partial_\sigma x^h(y; t) = a_\mu{}^\sigma\beta_\sigma{}^h$ which becomes on account of (3.4) and (3.6):

$$a_\mu{}^\sigma\partial_\sigma x^h(y; t) = \partial\beta_\mu{}^h/\partial t = \partial_\mu V^h.$$

From this result and (3.8) we get $\partial^2 x^h/\partial y^\lambda \partial t = a_\lambda{}^\sigma\partial_\sigma x^h$ and (c) is proved. Q.E.D.

We want to give an application. The space of (1, 1)-tensor fields $b_\lambda{}^\kappa$ satisfying (3.1) is a vector space over R if the case $b_\lambda{}^\kappa = c\delta_\lambda^\kappa$ is not excluded. Let us denote it by $\mathfrak{b}$. Let $l = \dim \mathfrak{b}$ and $h = \dim H^1(M)$. Then there exists a subspace $\mathfrak{b}_1$ of $\mathfrak{b}$ of dimension at least $l - h$ such that $b_\lambda{}^\sigma\partial_\sigma x^1$ is a gradient for each tensor field $b_\lambda{}^\kappa$ of $\mathfrak{b}_1$. There exists also a subspace $\mathfrak{b}_2$ of $\mathfrak{b}_1$ of dimension at least $l - 2h$ such that $b_\lambda{}^\sigma\partial_\sigma x^2$ is a gradient for each element $b_\lambda{}^\kappa$ of $\mathfrak{b}_2$ and so on. Thus, if $l - nh - 1 > 0$ holds, there exists a tensor field $b_\lambda{}^\kappa$ satisfying (ii) of Theorem 3.1. This proves:

THEOREM 3.2. *If $l = \dim \mathfrak{b}$ and $h = \dim H^1(M)$ satisfy $l - nh - 1 > 0$, then the immersion i_0M admits a nontrivial admissible deformation.*

4. Let M be a Veronese manifold immersed in a Euclidean space R^n in full. If $H_{\mu\lambda}{}^h$ is the second fundamental tensor at a point p of M, then for any vector u^h of R^n satisfying

$$\sum_h H_{\mu\lambda}{}^h u^h = 0,$$

we have [1]

$$u^h = t^\lambda B_\lambda{}^h.$$

As we have $n = m(m + 3)/2$ for the Veronese manifold, the above mentioned fact indicates that there exist among n symmetric $(m \times m)$ matrices $(H_{\mu\lambda}{}^h)$ just $m(m + 1)/2$ linearly independent symmetric matrices. This proves that the matrix $(a_\mu{}^\lambda)$ satisfying (1.4) is a scalar matrix. Thus we have

THEOREM 4.1. *A Veronese manifold immersed in a Euclidean space in full admits no nontrivial admissible deformation.*

This means that, when a Veronese manifold immersed in a Euclidean space in full undergoes a deformation other than a parallel displacement, a homothety, or their composition, then the Gauss image moves pointwise.

5. Let us consider an immersion i_0: $M \to R^n$ and an admissible deformation $M(t)$, $M(0) = i_0 M$, again. Let $H_{\mu\lambda}{}^h$ be the second fundamental tensor of $M(t)$ at $i(t)p$ where p is fixed. When μ and λ are given and h runs the range $1, \ldots, n$ we get components of a vector of R^n. There are m^2 such vectors and they span a vector space whose dimension we denote by d_p. Then d_p satisfies $d_p \leqslant m(m + 1)/2$ and $d_p \leqslant n - m$. In view of (1.7) we get:

THEOREM 5.1. *d_p is an invariant of any admissible deformation.*

BIBLIOGRAPHY

1. Y. Mutō, *Submanifolds of a Euclidean space with homothetic Gauss map*, J. Math. Soc. Japan (to appear).

2262-150, TOMIOKA-CHO, KANAZAWA-KU, YOKOHAMA 236, JAPAN