EXISTENCE OF CLOSED TIMELIKE GEODESICS IN LORENTZ SPACES

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Abstract. Certain classes of compact four-dimensional Lorentz spaces are shown to possess at least one closed timelike geodesic.

1. Introduction. The problem of existence of closed geodesics on a Riemannian manifold is one of the most fundamental questions in global differential geometry [1]. Existence of closed geodesics has been established for compact Riemannian manifolds, but no work has been done on establishing existence in compact Lorentz manifolds, possibly because it has been felt that existence proofs require a definite metric. In this paper a beginning is made toward extending existence theory of closed geodesics to compact Lorentz manifolds. Specifically, it is shown that in an important class of four-dimensional compact Lorentz manifolds—the ones with a covering space containing a compact Cauchy surface—closed timelike geodesics exist. My notation and conventions will be those of [2]. In particular, a spacetime \((M, g)\) is a \(C^\infty\) boundaryless four-dimensional connected Hausdorff manifold \(M\) with a \(C^\infty\) Lorentz metric \(g\). \((M, g)\) is time-orientable and orientable. All Cauchy surfaces \(S\) are boundaryless acausal imbedded submanifolds of \(M\) such that \(D(S) = M\); i.e., every inextendible causal curve in \(M\) intersects \(S\) exactly once.

2. The Theorem. Let \((M, g)\) be a compact spacetime with a covering space containing a compact Cauchy surface. Then \((M, g)\) contains a closed timelike geodesic.

Proof. Let \((\tilde{M}, \tilde{g})\) denote the covering space to \((M, g)\) which contains a compact Cauchy surface. Geroch [3, p. 444] (see also [2, p. 212]) has shown that \(\tilde{M}\) is homeomorphic to \(\tilde{S} \times \mathbb{R}\), where \(\tilde{S}\) is the Cauchy surface, and that all Cauchy surfaces in \((\tilde{M}, \tilde{g})\) are compact. Since \((M, g)\) is a compact spacetime, it must contain a closed timelike curve \(\gamma\) [2, p. 189]. Let \(\tilde{\gamma} \subset \tilde{M}\) be...
a timelike curve segment which is mapped one-to-one and onto $\gamma$ by the
covering map $f$. The closure of $\tilde{\gamma}$ has two endpoints, $\tilde{p}'$ and $\tilde{p}''$, both of which
are mapped into the same point of $\gamma$ by the covering map. By the compact-
ness of $\tilde{S}$, the fact that $\tilde{M}$ is homeomorphic to $\tilde{S} \times \mathbb{R}$, and the properties of
the covering map, there exist disjoint compact Cauchy surfaces $\tilde{S}'$ and $\tilde{S}''$
such that $\tilde{p}' \in \tilde{S}'$, $\tilde{p}'' \in \tilde{S}''$, and $\tilde{S}'$ is isometric to $\tilde{S}''$. (Pick any compact
Cauchy surface $\tilde{S}'$ through $\tilde{p}'$, and use the covering map to construct its
image $\tilde{S}''$ through $\tilde{p}''$. The Cauchy surfaces $\tilde{S}'$ and $\tilde{S}''$ will be disjoint by the
properties of the covering map.) The distance function $d(\bar{u}', \bar{v}'')$ for all points
$\bar{u}' \in \tilde{S}'$ and $\bar{v}'' \in \tilde{S}''$ is finite and continuous in $\bar{u}'$ and $\bar{v}''$ by Lemma 6.7.3 of
[2, p. 215]; in particular it is finite and continuous for corresponding points
$\tilde{p}', \tilde{p}''$ of $\tilde{S}'$ and $\tilde{S}''$, respectively. Since $d(\tilde{p}', \tilde{p}'')$ is a continuous function on
the compact set $\tilde{S}''$, it attains its maximum value for the points $\tilde{q}', \tilde{q}''$ say. By
Proposition 6.7.1 of [2, p. 213] and the argument on p. 216 of [2], there exists
a timelike geodesic $\alpha$ of length $d(\tilde{q}', \tilde{q}'')$ between $\tilde{q}'$ and $\tilde{q}''$. The geodesic $\alpha$
projects into a closed timelike curve $\alpha$ in $(M, g)$, and $\alpha$ is a geodesic
everywhere except possibly at $\tilde{f}(\tilde{q}') = \tilde{f}(\tilde{q}'') = q$, where the tangent to $\alpha$ might
not be defined. However, $\alpha$ must be a geodesic with well-defined tangent at $q$
also, for otherwise by Proposition 4.5.3 of [2, p. 105] we could deform $\alpha$ into a
closed timelike curve whose cover in $(\tilde{M}, \tilde{g})$ would have greater length from
$\tilde{S}'$ to $\tilde{S}''$ than $\tilde{\alpha}$. But this is impossible since $\alpha$ has maximal length among all
timelike curves from a point in $\tilde{S}'$ to the corresponding point in $\tilde{S}''$. Thus $\alpha$
is a closed timelike geodesic in $(M, g)$.

3. Discussion. The above theorem can be extended without much difficulty
to any dimension greater than or equal to 2, since all the necessary lemmas
from [2] can easily be generalized to any dimension $\geq 2$. I have restricted the
proof given above to four dimensions in order to simplify the proof.

The restriction to compact Lorentz manifolds with globally hyperbolic
covering spaces is unfortunately very severe. Such a restriction is necessary in
order to use without modification the distance function theory for Lorentz
manifolds developed by Hawking [2]. However, the above theorem does
apply to most simple examples of Lorentz manifolds. The restriction to
nonsimply connected compact Lorentz manifolds is not strong, at least in
dimensions, since in four dimensions all compact Lorentz manifolds are
not simply connected [2, p. 190]. It is hoped that the ideas used in the proof
of the above theorem could suggest a proof for general Lorentz manifolds.

It should be mentioned that no compact Lorentz manifold is of any
physical interest, since all such manifolds have closed timelike curves ([2, p.
189], [4, p. 38], [5, p. 783]).

REFERENCES

1. W. Klingenberg. Lectures on closed geodesics, Grundlehren der Mathematischen Wissen-


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