THE MILNOR SIGNATURES OF COMPOUND KNOTS

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Abstract. The Milnor signatures of a classical knot are related to those of its companions.

1. We shall work throughout in the smooth or piecewise-linear category. A knot $k$ is a circle $S^1$ embedded in the 3-sphere $S^3$. A regular neighbourhood $V$ of $k$ is a solid torus. A longitude of $\partial V$ is a circle embedded in $\partial V$ which is homologous to $k$ in $V$, and null-homologous in the closed complement of $V$. We assume that all knots and longitudes are oriented.

Let $F$ be a solid torus unknotted in $S^3$ and containing a knot $l^*$, and let $f$ be a faithful map from $T$ onto $V$ (that is, a homeomorphism which takes a longitude of $\partial F$ onto a longitude of $\partial V$). If $l^*$ represents $n \in \mathbb{Z} = H_1(T)$, then $l = f(l^*)$ is homologous to $nk$ in $V$. Proofs will be presented as though $n$ were positive, but with trivial adjustments in notation they are valid for all $n$.

Let $\Delta_l(t)$ be the Alexander polynomial of the knot $l$, and $\Delta_k(t)$, $\Delta_{nk}(t)$ those of $k$, $l^*$ respectively. It is a result of Seifert [S] that $\Delta_l(t) = \Delta_k(t^n) \cdot \Delta_{nk}(t)$.

If $p(t)$ is a symmetric, quadratic factor of $\Delta_k(t)$, irreducible over the real numbers, then we can write $p(t)$ in the form $t^{-1} - 2 \cos \theta + t$, $0 < \theta < \pi$. Milnor [M1] has defined a signature $\sigma_\theta(k)$ which is an invariant of $k$. Let $p(t^n) = p_1(t) \cdots p_n(t)$, where each $p_j(t)$ is symmetric, quadratic and irreducible over the real numbers, and let $\exp(i\theta_j)$ be the root of $p_j(t)$ which is also an $n$th root of $\exp(i\theta)$, where $0 < |\theta_j| < \pi$.

Theorem. $\sigma_{|\theta_j|}(l) = \sigma_{|\theta_j|}(l^*) + \sigma_\theta(k)\text{sign}(n \sin \theta_j)$, where $\sigma_{|\theta_j|}(l^*) = 0$ if $\exp(i\theta_j)$ is not a root of $\Delta_{nk}(t)$.

If $\exp(i\theta)$ is a root of $\Delta_{nk}(t)$ but not of $\Delta_k(t^n)$, then $\sigma_{\theta}(l) = \sigma_{\theta}(l^*)$.

Corollary.

$$\sigma(l) = \sigma(l^*) \quad \text{if } n \text{ is even},$$

$$= \sigma(l^*) + \sigma(k) \quad \text{if } n \text{ is odd}.$$
map $\pi_1(K) \to H_1(K) \cong \pi_1$. Then $H_1(\tilde{K})$ is a finitely-generated module over $\Lambda = \mathbb{Z}[t, t^{-1}]$, and there is a Blanchfield duality pairing

$$\langle , \rangle : H_1(\tilde{K}) \times H_1(\tilde{K}) \to \Lambda_0 / \Lambda,$$

where $\Lambda_0$ is the field of fractions of $\Lambda$. This pairing is Hermitian with respect to the conjugation defined by $t \mapsto t^{-1}$. It is also nonsingular.

Set $\Gamma = \mathbb{R}[t, t^{-1}]$, and pass to real coefficients: then we obtain a pairing

$$\langle , \rangle : H_1(\tilde{K}; \mathbb{R}) \times H_1(\tilde{K}; \mathbb{R}) \to \mathbb{R}/\mathbb{R}.$$

Let $p(t)$ be a prime in $\Gamma$ dividing $\Delta_k(t)$; and let $V_p$ denote the $p(t)$-primary component of $H_1(\tilde{K}; \mathbb{R})$. As in [K], $V_p$ is orthogonal to $V_q$ unless $(p(t)) = (q(t^{-1}))$. Moreover, if $p(t) = p(t^{-1})$, then $V_p$ can be written as an orthogonal direct sum $V_p^1 \oplus \cdots \oplus V_p^m$, with $V_p^i$ a free module over $\Gamma/(p^r)$. Let $(x)$ denote the image in $H^*_p = V_p^*/pV_p^*$ of $x$ in $V_p^*$, if $x, y \in V_p^*$, then we can define $[(x), (y)]_p^* = \langle p(t)^{-1}x, y \rangle$.

Let $\varphi : \Gamma \to \Gamma/(p)$ be the quotient map; then defining $((x), (y))_p = \varphi(z)$, where $[(x), (y)]_p^* = z/p$, makes $H^*_p$ into an Hermitian space over the field $\Gamma/(p) \cong \mathbb{C}$. Conjugation coincides with complex conjugation, as the roots of $p(t)$ lie on the complex unit circle. Let $\sigma_p(k)$ be the signature of the corresponding quadratic space, and let $\sigma_p(k)$ be the sum over odd $r$ of the $\sigma_p(k)$.

It is shown in [K] that $\sigma_p(k) = \sigma_0(k)$, where $p(t) = t^{-1} - 2 \cos \theta + t$, $0 < \theta < \pi$.

In passing, note that $\sigma_p(k)$ is an invariant of the cobordism class of $k$ (see [M1]); $\sigma_p(k)$ is an invariant of $k$, but not of its cobordism class [L]. As Milnor points out [M2], for $r$ even the corresponding quadratic space is hyperbolic, and so $\sigma_p(k) = 0$.

3. Let $N$ be a regular neighbourhood in $T$ of $I^*$. $L^*$ the closed complement of $N$ in $S^3$, $L' \cup K$ the closed complement of $f(N)$ in $V$, and $L$ the closed complement of $f(N)$ in $S^3$. Then $L = L' \cup K$, and $L' \cap K = \partial V$. Passing to the infinite cyclic cover of $L$, $\tilde{L} = \tilde{L}' \cup \tilde{K}_1 \cup \cdots \cup \tilde{K}_n$, where the $\tilde{K}_r$ are disjoint copies of $\tilde{K}$, and $\tilde{L}' \cap \tilde{K}_r = S^1 \times \mathbb{R}$. We can number the $\tilde{K}_r$ so that the action of $(t:) \in L$ is given by $t\tilde{K}_r = \tilde{K}_{r+1}$, working modulo $n$.

It is implicit in the work of Seifert [S] that $H_1(\tilde{L})$ splits as a direct sum of $\Lambda$-modules, $H_1(\tilde{L}^*) \oplus H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_n)$. Furthermore, if $M(t)$ is a presentation matrix for $H_1(\tilde{K})$, then $M(t^n)$ is a presentation matrix for $H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_n)$: this too is easily deduced from [S; p. 32].

From the definition of the Blanchfield duality pairing [B], it is clear that the direct sum above is orthogonal, and that $H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_n)$ splits as an orthogonal direct sum of $\mathbb{Z}$-modules $H_1(\tilde{K}_1) \oplus \cdots \oplus H_1(\tilde{K}_n)$.

4. Let $p(t) = t^{-1} - 2 \cos \theta + t$, $0 < \theta < \pi$, be an irreducible factor of $\Delta_k(t)$, and let $p(t^n) = p_1(t) \cdots p_n(t)$ where $p_i(t) = t^{-1} - 2 \cos \theta_i + t$, $0 < \theta_i < \pi$. Let $\tau = \exp(i\theta)$, and let $\tau_r = \exp(i\theta_r)$ be the root of $p_i(t)$ which is also an $n$th root of $\tau$. Write $p^r(t) = p(t^n)/p_i(t)$. 

Recall that $V_p$ is the $p(t)$-primary component of $H_1(\tilde{K}; R)$. If we identify $H_1(\tilde{K}; R)$ with $H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_s; R)$, as a vector space, then clearly $(p'(t))^N V_p$ is contained in $V_p$, the $p(t)$-primary component of $H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_s; R)$, for large $N$. Indeed, by considering a diagonal presentation matrix $M(t)$ for $H_1(K; R)$, and passing to $M(t^n)$, it is clear that $(p'(t))^N V_p = V_p$.

Consider $x, y \in V_p$ as elements of $H_1(\tilde{K}; R)$; then $\langle x, y \rangle = \mu(t)/(p(t))^m$ say. Regarding $x, y$ as elements of $H_1(\tilde{K}_1 \cup \cdots \cup \tilde{K}_s; R)$, it follows from the definition of the duality pairing that $\langle x, y \rangle = \mu(t^n)/(p(t^n))^m$. Thus if $V_p = V_p^1 \oplus \cdots \oplus V_p^m$, an orthogonal direct sum in $H_1(\tilde{K}; R)$, we can take $V_p^s = (p'(t))^N V_p^s$ to obtain an orthogonal direct sum $V_p = V_p^1 \oplus \cdots \oplus V_p^m$.

Let $x', y' \in H_1^s$, and choose $x, y \in V_p$ so that $x' = ((p'(t))^{N-1}x), y' = ((p'(t))^Ny)$. Then

$$[x', y']_p = \left\langle p(t)^{t-1}(p'(t))^N x, (p'(t))^N y \right\rangle$$
$$= (p'(t))^{2N-2s+1} \left\langle p(t^n)^{s-1} x, y \right\rangle$$
$$= (p'(t))^{2N-2s+1} \mu(t^n)/p(t^n)$$

where regarding $x, y$ as elements of $H_1(\tilde{K}; R)$, the Blanchfield pairing of $k$ gives $\langle p(t^{t-1})x, y \rangle = \mu(t)/p(t)$. Thus

$$[x', y']_p = (p'(t))^{2N-2s+1} \mu(t^n)/p(t),$$

and so

$$(x', y')_p = (p'(t))^s \mu(t).$$

Of course, if we regard $x, y$ as elements of $H_1(\tilde{K}; R)$, then in the Hermitian space $H_p^s$ we have $((x), (y))_p = \mu(t)$. Thus it only remains for us to evaluate $p'(t)\mu(t)$. Using L'Hôpital's rule, it is easy to see that

$$p'(t) = \lim_{t \to t} \frac{p(t)}{p'(t)} = n \frac{\Im(t)}{\Im(t')} = n \frac{\sin \theta}{\sin \theta'},$$

where $\Im(z)$ is the imaginary part of $z$.

Thus if $s$ is odd, $H_p^s$ contributes $a_s(k) \text{ sign } (n \sin \theta)$ to the signature of $l$; and if $s$ is even, all the corresponding signatures are zero. This proves the theorem.

The corollary follows easily by considering the distribution of the $\tau_i$ around the unit circle.

References


