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SHORTER NOTES

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A CLASSICAL VARIATIONAL PRINCIPLE FOR PERIODIC HAMILTONIAN TRAJECTORIES[1]

FRANK H. CLARKE

ABSTRACT. Using only classical theorems of the calculus of variations, the existence of periodic solutions to Hamilton's equations on a given convex energy surface is proved.

Let H: $R^n \times R^n \to R$ be a given C^1 function (the *Hamiltonian*). Consider the Hamiltonian system of equations

$$-\dot{p} = H_q(q, p), \qquad \dot{q} = H_p(q, p) \tag{1}$$

where q and p are absolutely continuous functions mapping $[0, T]$ $(T > 0)$ to R^n. The problem we consider is the possible existence of a solution (q, p) of these equations satisfying $q(0) = q(T)$, $p(0) = p(T)$, and also $H(q, p) = c$, where c is a given constant. (The reader will recall that any solution to (1) automatically lies on a level surface of H.) We call such a solution periodic, and we say it lies on $H^{-1}(c)$.

Of course the existence of (q, p) can only be guaranteed under suitable conditions on H. Recently P. H. Rabinowitz [3] and A. Weinstein [4], [5] have specified such conditions, using recent and sophisticated techniques. In so doing, it is not the case that the periodic solution arises as the solution (or even critical point) of a variational problem; i.e. no variational principle is found. In [2] the author gives a variational principle for such trajectories, one that involves a "differential inclusion" problem of optimal control and new necessary conditions for such problems. In the present article, we use a different approach. While the results so obtained are somewhat less general (for example, in contrast to [2], $H^{-1}(c)$ must be smooth and *strictly* convex),

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we believe it is interesting that it is possible to work entirely within the context of the classical calculus of variations (we employ the Legendre transform, the Tonelli existence theorem, and Hilbert's multiplier rule for isoperimetric problems). The theorem proven below is a version of Theorem 1.1 of [3] (which allows the star-shaped case) and is more general than that of [5].

THEOREM. *Let $H^{-1}(c)$ be the boundary of a compact strictly convex set containing 0 in its interior, and suppose $\nabla H \neq 0$ on $H^{-1}(c)$. Then, for some $T > 0$, there is a periodic solution of* (1) *on $H^{-1}(c)$.*

PROOF. Define $\bar{H}(q, p)$ to be λ^2, where λ is the unique scalar > 0 such that $(q, p)/\lambda$ lies in $H^{-1}(c)$ (we set $\bar{H}(0, 0) = 0$). Then $\bar{H}$ is C^1, and $\bar{H}^{-1}(1) = H^{-1}(c)$. Further, for any (q, p) in $H^{-1}(c)$, $\nabla \bar{H}(q, p)$ is a nonzero multiple of $\nabla H(q, p)$, and it follows that any periodic solution to (1) for $\bar{H}$ gives rise to a periodic solution to (1) for H after a change of the time variable. The upshot of the foregoing (which is the preliminary step introduced by Rabinowitz [3, Lemma 1.5]) is the conclusion: *it suffices to prove the theorem for $H = \bar{H}$ and $c = 1$.* This we now do.

We set L equal to the Legendre transform of $\bar{H}$:

$$L(v, w) = (v, w) \cdot (a, b) - \bar{H}(a, b)$$

where (a, b) is the unique solution of $(v, w) = \nabla \bar{H}(a, b)$. Equivalently,

$$L(v, w) = \max_{a,b} \{(v, w) \cdot (a, b) - \bar{H}(a, b)\}.$$

Since $\bar{H}$ is strictly convex and grows quadratically, it follows that L is C^1 (and of course convex). The estimate $L(v, w) \geqslant \varepsilon|(v, w)|^2$, for some $\varepsilon > 0$, is elementary. Consider now the following isoperimetric problem:

$$\text{minimize} \int_0^1 L(-\dot{p}, \dot{q})\, dt$$

subject to $\int_0^1 p \cdot \dot{q}\, dt = 1$, $q(0) = q(1) = 0$, $p(0) = p(1) = 0$. Then a solution $(\tilde{q}, \tilde{p})$ exists by the Tonelli existence theorem (see for example [1]), and by the multiplier rule [1] there exist λ_0, λ_1 not both zero such that $(\tilde{q}, \tilde{p})$ is an extremal for $\lambda_0 L(-\dot{p}, \dot{q}) + \lambda_1 p \cdot \dot{q}$. The Euler-Lagrange equation in undifferentiated form yields

$$\lambda_0 L_w(-\dot{\tilde{p}}, \dot{\tilde{q}}) + \lambda_1 \tilde{p} = c_1, \qquad -\lambda_0 L_v(-\dot{\tilde{p}}, \dot{\tilde{q}}) = c_2 + \lambda_1 \tilde{q}.$$

It follows that if $\lambda_0 = 0$, then $\tilde{p}$ is constant, which is not possible in view of the isoperimetric constraint. Hence we may assume $\lambda_0 = 1$. Now if $\lambda_1 = 0$, then $\nabla L(-\dot{\tilde{p}}, \dot{\tilde{q}})$ is constant, which implies that $(\dot{\tilde{q}}, \dot{\tilde{p}})$ is constant (since L is strictly convex), which is not possible since $\tilde{p}$ is nonconstant and periodic. So the functions $\hat{q} = -\lambda_1 \tilde{q} - c_2$ and $\hat{p} = c_1 - \lambda_1 \tilde{p}$ satisfy $(\hat{q}, \hat{p}) = \nabla L(\dot{\hat{p}}/\lambda_1, -\dot{\hat{q}}/\lambda_1)$. This is equivalent to $-(-\dot{\hat{p}}, \dot{\hat{q}})/\lambda_1 = \nabla \bar{H}(\hat{q}, \hat{p})$ by Legendre duality, and it follows that $(\hat{q}, \hat{p})$ lies on a surface $\bar{H}^{-1}(b)$ for some positive b. We now define

$$q(t) = \hat{q}(-t/\lambda_1)/b^{1/2}, \qquad p(t) = \hat{p}(-t/\lambda_1)b^{1/2}$$

if λ_1 is negative, and otherwise we set

$$q(t) = \hat{q}(1 - t/\lambda_1)/b^{1/2}, \qquad p(t) = \hat{p}(1 - t/\lambda_1)b^{1/2}.$$

If we observe that by construction $\overline{H}$ is positively homogeneous of order 2, and $\nabla\overline{H}$ of order 1, it then follows easily that (q, p) satisfies (1) and is periodic on the interval $[0,|\lambda_1|]$, and that (q, p) lies on $\overline{H}^{-1}(1)$. Q.E.D.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BRITISH COLUMBIA, CANADA V6T 1W5