RATIONAL SURFACES WITH TOO MANY VECTOR FIELDS

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Abstract. A method is given for constructing smooth rational surfaces with nonreduced automorphism groups, by a sequence of blow-ups of the projective plane. The technique works in all positive characteristics.

1. Introduction. The vector space $V_X = H^0(X, \Theta_X)$ of regular vector fields on a surface $X$ can be identified with the tangent space to the group scheme $\text{Aut } X$ at its identity element. Thus a surface $X$ has a nonreduced automorphism group scheme iff $\dim V_X > \dim \text{Aut } X$. We wish to describe a method for constructing smooth rational surfaces with this property. Since group schemes in characteristic zero are smooth [1, p. 101], this will be possible only for positive characteristics. (In the language of deformation theory, $V_X$ may also be identified with the set of infinitesimal automorphisms of $X$ parameterized by the scheme $\text{Spec } k[\![t]\!]/(t^2)$. If $\dim V_X > \dim \text{Aut } X$, then $X$ has "obstructed" infinitesimal automorphisms, that is, infinitesimal automorphisms which do not extend to an algebraic family of automorphisms.)

If $X$ is a smooth surface, and $Y$ is the blow-up of $X$ at a smooth point $P$, then the relationship between $\text{Aut } X$ and $\text{Aut } Y$ can be summarized in two rules.

Lemma (1.1). The identity component $\text{Aut}^0 Y$ is a closed subgroup of $\text{Aut } X$, whose support is the identity component of the subgroup of automorphisms which fix $P$.

Lemma (1.2). The tangent space $V_Y$ is the subspace of $V_X$ consisting of the vector fields which vanish at $P$.

Lemma (1.2) is the first step in the proof of Lemma (1.1). It follows from Lemma (3.1) below. If we let $C$ denote the closed subscheme of $\text{Aut } X$ representing the functor $C: T \mapsto \{ T\text{-Automorphisms of } X \times T \text{ fixing the closed subscheme } P \times T \}$, then it is not difficult to show that $\text{Aut}^0 Y$ is isomorphic to the identity component $C^0$. Details may be found in [3].

Thus our strategy is to blow up at a point which is fixed by too few automorphisms, or, equivalently, at which too many vector fields vanish.

The projective plane has a reduced automorphism group, namely $\text{PGL}(2)$. We will make a series of blowings-up of the plane, each time at a point
indefinitely near the previous point, that is, on the new exceptional divisor. It is convenient to use multi-projective coordinates to describe such a procedure.

Let $B_0$ denote the projective plane $\mathbb{P}^2$ with coordinates $(X_0, X_1, X_2)$.

Let $P_0$ be the point $(1, 0, 0)$. We can also take inhomogeneous coordinates $x_1 = X_1/X_0$ and $x_2 = X_2/X_0$ at $P_0$. The blow-up $B_1$ of the surface $B_0$ at the point $P_0$ can be viewed as the subvariety of $B_0 \times \mathbb{P}^1$ consisting of all points $(X_0, X_1, X_2; Y_0, Y_1)$ for which $X_1 Y_1 = X_2 Y_0$. In inhomogeneous coordinates, with $y_1 = Y_1/Y_0$, this gives the familiar local equation $x_1 y_1 = x_2$.

Choose a point $P_1 = (1, 0, 0; 1, \lambda_1)$ on the exceptional divisor. The blow-up of $B_1$ at this point consists of all points of $B_1 \times \mathbb{P}^1$ of the form $(X_0, X_1, X_2; Y_0, Y_1; Z_0, Z_1)$ satisfying the equation

$$X_1 Y_0 Z_1 = (Y_1 - \lambda_1 Y_0) Z_0 X_0.$$

Once more, taking the inhomogeneous coordinate $y_2 = Z_1/Z_0$ gives the local equation $x_1 y_2 = y_1 - \lambda_1$.

Similarly, for any choice of $\lambda_1, \ldots, \lambda_k$, we can take a point $P_k = (1, 0, 0; 1, \lambda_1; \ldots; 1, \lambda_k)$ on $B_k$ and blow up to get a surface $B_{k+1}$ defined in $B_k \times \mathbb{P}^1$ as the set of all $(X_0, X_1, X_2; Y_0, Y_1; \cdots; V_0, V_1; W_0, W_1)$ for which

$$X_1 V_0 W_1 = (V_1 - \lambda_k V_0) W_0 X_0,$$

and with $y_{k+1} = W_1/W_0$ and $y_k = V_1/V_0$, this gives the local equation $x_1 y_{k+1} = y_k - \lambda_k$. With this notation, we will investigate how $\text{Aut } B_{k+1}$ and $\dim H^0(B_{k+1}, \Theta_{B_{k+1}})$ depend on the choice of $\lambda_1, \ldots, \lambda_k$. We can take $\lambda_1 = 0$ without loss of generality.

2. Calculation of automorphisms. We wish to compute the dimension of $\text{Aut } B_{k+1}$. By Lemma (1.1), it suffices to determine which automorphisms of $B_k$ fix the point $P_k$. Applying the lemma repeatedly, we can view an element $\alpha$ of the identity component $\text{Aut}_0 B_k$ as an automorphism of $\mathbb{P}^2$. If $\alpha$ fixes $P_k$, it must send each curve passing through $P_k$ to another curve with the same property. This condition can be stated on $\mathbb{P}^2$: we ask that $\alpha$ send each curve whose proper transform on $B_k$ passes through $P_k$ to another such curve.

Given $\lambda_1, \ldots, \lambda_k$, we put

$$F_k(T) = \lambda_2 T^2 + \cdots + \lambda_k T^k.$$

If $S_k$ denotes Spec $k[t]/(t^{k+1})$, we define

$$f_k: S_k \rightarrow \mathbb{P}^2$$

by

$$t \leftrightarrow x_1, \quad F_k(t) \leftrightarrow x_2.$$

With this notation, a straightforward calculation proves the following lemma.

**Lemma (2.1).** Let $C$ be a smooth curve on $\mathbb{P}^2$. Then the proper transform of $C$ on $B_k$ passes through $P_k$ iff $f_k$ factors through $C$. 

Thus to fix $P_k$, an automorphism of $\mathbb{P}^2$ must commute with $f_k$ up to an automorphism of $S_k$.

We recall that every automorphism $\alpha$ of $\mathbb{P}^2$ can be written as a linear map

$$
\begin{align*}
\alpha: & \left[ \begin{array}{c} X_0 \\ X_1 \\ X_2 
\end{array} \right] \rightarrow \left[ \begin{array}{ccc} aX_0 & + & bX_1 & + & cX_2 \\ dX_0 & + & eX_1 & + & fX_2 \\ gX_0 & + & hX_1 & + & iX_2 
\end{array} \right],
\end{align*}
$$

and that two linear maps give the same automorphism if they differ by a scalar. We wish to determine the conditions imposed on the coefficients of the matrix

$$
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix}
$$

when we require $\alpha$ to fix $P_k$.

To fix $P_0$ at all, we must have $d = g = 0$. If $\alpha$ also fixes $P_1$, it must fix the line $X_2 = 0$, and so $h$ is likewise zero. Then the matrix is invertible and upper triangular, and its diagonal entries $a$, $e$, and $i$ are all nonzero. We can now eliminate the ambiguity about scalars by choosing $a = 1$.

Then $\alpha$ commutes with $f_k$ up to an automorphism of $S_k$ iff

$$
F_k \left( \frac{eu + fF_k(u)}{1 + bu + cF_k(u)} \right) \equiv \frac{iF_k(u)}{1 + bu + cF_k(u)} \mod u^{k+1}. \quad (2.2)
$$

AN EXAMPLE. Rather than describe the conditions imposed on the entries $b$, $c$, $e$, $f$, and $i$ by equation (2.2) in general, we would like to work out a specific example, which gives us in each positive characteristic one of our rational surfaces with nonreduced automorphism group.

Let the characteristic of the ground field be $p$. Then we will take $\lambda_1 = \lambda_2 = \cdots = \lambda_{2p} = 0$, $\lambda_{2p+1} = \lambda \neq 0$, $\lambda_{2p+2} = \cdots = \lambda_{3p} = 0$, and $\lambda_{3p+1} = \mu \neq 0$.

Then the condition that $\alpha$ fix $P_{3p+1}$ is

$$
\lambda \left( \frac{eu + f\lambda u^{2p+1} + f\mu u^{3p+1}}{1 + bu + c\lambda u^{2p+1} + c\mu u^{3p+1}} \right)^{2p+1} + \mu \left( \frac{eu + f\lambda u^{2p+1} + f\mu u^{3p+1}}{1 + bu + c\lambda u^{2p+1} + c\mu u^{3p+1}} \right)^{3p+1} \equiv \frac{i\lambda u^{2p+1} + i\mu u^{3p+1}}{1 + bu + c\lambda u^{2p+1} + c\mu u^{3p+1}} \mod u^{3p+2}.
$$

The congruence breaks into the two equations:

$$
i\lambda = \lambda e^{2p+1}
$$

and

$$
3b^2i\lambda + u\mu = u\mu e^{2p+1} + b^2e^{2p+1}\lambda;
$$
solving simultaneously, the latter yields
\[\mu e^{2p+1}(1 - e^p) = \lambda e^{2p+1}b^p(1 - 3).\]

Clearly these give two conditions on the coefficients, and the automor-
phism group thus has dimension three; we will later see that this choice of \(\lambda\)'s
imposes only one condition on the tangent fields, giving a tangent space of
dimension four.

3. Calculation of vector fields. Given our sequence of surfaces
\(B_1, \ldots, B_{k+1}\), we wish to calculate the dimension of \(V_{b_{k+1}} = H^0(B_{k+1}, \Theta_{b_{k+1}})\)
by determining inductively which vector fields on \(B_j\) vanish at \(P_j\). To do so,
we will first take a vector field vanishing at \(P_{j-1}\) and find an expression for
the vector field to which it lifts on \(B_j\).

We can take \(x_j\) and \(y_{j-1} - \lambda_{j-1}\) as parameters at the point \(P_{j-1}\). Any vector
field \(\theta_{j-1}\) on \(B_{j-1}\) can then be written in the form \(f d/dx + g dy\).
If it lifts to a vector field \(\theta_j\) on \(B_j\), we wish to find an expression for \(\theta_j\) in the form
\[f d/dx + g dy\j.

**Lemma (3.1).** The vector field \(\theta_{j-1}\) lifts to a regular vector field \(\theta_j\) on \(B_j\) iff it
vanishes at \(P_{j-1}\), and then at \(P_j\) we can write
\[\theta_j = f \frac{d}{dx} + g \frac{d}{dy},\]
with \(g_j = \frac{1}{x_j}(g - fy_j)\).

**Proof.** Locally at \(P_{j-1}\), the surface \(B_{j-1}\) is an open subset of
\(\text{Spec } k[x_j, y_{j-1}]\), and the blowing-up corresponds to the ring homomorphism
\[k[x_j, y_{j-1}] \to k[x_j, y_{j-1}, y_j]/(y_j - \lambda_{j-1} - x_j y_j)\).

We may view the vector fields as derivations on these rings, and we will let
\(D_w\) denote \(d/dw\) on \(k[x_j, y_{j-1}]\), and \(\Delta_w\) denote \(d/dw\) on \(k[x_j, y_{j-1}, y_j]/(y_j - \lambda_{j-1} - x_j y_j)\).

Then
\[D_{x_j} = \Delta_{x_j} \circ \phi - \frac{y_j}{x_j} \Delta_{y_j} \circ \phi \quad \text{and} \quad D_{y_j} = \frac{1}{x_j} \Delta_{y_j} \circ \phi.

We conclude that \(f d/dx + g dy/dy\) must lift to the vector field
\[f \frac{d}{dx} + \frac{1}{x_j}(g - fy_j) \frac{d}{dy_j}.

A priori, a vector field of this form has coefficients in the function field
\(k(x_j, y_j)\); for \(\theta_j\) to be regular on our open subset of \(B_j\), \(x_j\) must divide \(g - fy_j\).
This occurs iff \(g\) has no constant term, that is, iff \(g\) vanishes at \(P_{j-1}\). By
symmetry, \(\theta_j\) is regular on all of \(B_j\) iff both \(f\) and \(g\) vanish at \(P_{j-1}\). That is,
\(\theta_{j-1}\) must vanish at \(P_{j-1}\). This proves the lemma.

(Since the question is local in the étale topology, this also proves Lemma
(1.2).)
We suppose now that $\theta_{j-1}$ vanishes at $P_{j-1}$, and so lifts to $\theta_j$ on $B_j$. When does $\theta_j$ vanish at $P_j$? The function $f$ already vanishes at $P_{j-1}$, and therefore vanishes on the entire exceptional divisor over $P_{j-1}$. It remains to examine $g_j = (1/x_1)(g - f_j)$. If we break $g - f_j$ into a sum of homogeneous parts with respect to $x_1$,

$$g - f_j = H_0 + H_1x_1 + H_2x_1^2 + \ldots,$$

we see that $H_0$ must vanish at $P_j$, and that $g_j$ vanishes at $P_j$ iff $H_1$ does.

Thus the vector field $\theta_j$ vanishes at $P_j$ iff the coefficient $H_1$ of the part of $x_1$-degree one in $g - f_j$ does.

Returning to $B_0$, we recall that $H^0(\mathbb{P}^2, \Theta_p)$ is spanned by the fields $X_i d/dX_j$, with the single relation

$$X_0 \frac{d}{dX_0} + X_1 \frac{d}{dX_1} + X_2 \frac{d}{dX_2} = 0.$$

This relation permits us to express $X_i d/dX_0$ as

$$X_i \frac{d}{dX_0} = \frac{X_i}{X_0} X_0 \frac{d}{dX_0} = -\frac{X_i}{X_0} \frac{d}{dX_1} - \frac{X_i}{X_0} \frac{d}{dX_2}.$$

Since

$$X_0 \frac{d}{dX_j} = \frac{d}{d(X_j/X_0)},$$

we can write a basis in parameters at $(1, 0, 0)$ in the form

$$\alpha = X_0 \frac{d}{dX_0} = -x_1 \frac{d}{dx_1} - x_2 \frac{d}{dx_2},$$

$$\beta = X_1 \frac{d}{dX_0} = -x_1^2 \frac{d}{dx_1} - x_1x_2 \frac{d}{dx_2},$$

$$\gamma = X_2 \frac{d}{dX_0} = -x_1x_2 \frac{d}{dx_1} - x_2^2 \frac{d}{dx_2},$$

$$\delta = X_0 \frac{d}{dX_1} = \frac{d}{dx_1},$$

$$\epsilon = X_1 \frac{d}{dX_1} = x_1 \frac{d}{dx_1},$$

$$\zeta = X_2 \frac{d}{dX_1} = x_2 \frac{d}{dx_1},$$

$$\eta = X_0 \frac{d}{dX_2} = \frac{d}{dx_2},$$

$$\theta = X_1 \frac{d}{dX_2} = x_1 \frac{d}{dx_2}.$$

The fields $\alpha, \beta, \gamma, \epsilon, \zeta, \eta,$ and $\theta$ vanish at $(1, 0, 0)$ and lift to $B_1$; all of these but $\theta$ vanish at $P_1$, and lift to $B_2$.\n
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Lemma (3.1) allows us to compute the form a vector field on $B_{j-1}$ takes on $B_j$; this recursive procedure also yields an explicit equation. If a vector field has the form \( f/dx_1 + g_k d/dy_k \) at $P_k$ for each $k$, then the $g_k$ are related by the formula $g_k = (1/x_1)(g_{k-1} - \tilde{f}_k)$, and so

$$g_k = \frac{g_1}{x_1^{k-1}} - f \sum_{r=2}^{k} \frac{y_1}{x_1^{k-r+1}}. \tag{3.2}$$

Using the defining equation for $y_j$, $y_j x_1 = y_{j-1} - \lambda_{j-1}$, we can compute that

$$\frac{y_j}{x_1} = y_{j+n} + \sum_{m=0}^{n-1} \frac{\lambda_{m+j}}{x_1^{n-m}}. \tag{3.3}$$

It then follows that

$$\sum_{r=2}^{k} \frac{y_1}{x_1^{k-r+1}} = (k - 1) \frac{y_k}{x_1} + \sum_{m=0}^{k-1} (m - 1) \frac{\lambda_m}{x_1^{k-m+1}}. \tag{3.4}$$

Since we are interested in $H_1 x_1$, the term of $x_1$-degree one in $g_{k-1} - \tilde{f}_k$, we combine formulae (3.2) and (3.4) in the single equation

$$g_{k-1} - \tilde{f}_k = x_1 g_k = \frac{g_1}{x_1^{k-2}} - (k - 1) \tilde{f}_k - f \sum_{m=1}^{k-1} (m - 1) \frac{\lambda_m}{x_1^{k-m}}. \tag{3.5}$$

Then by inspection, using (3.3) in the form

$$y_1 = x_1^{k-1} \left( y_k + \sum_{m=1}^{k-2} \frac{\lambda_{m+1}}{x_1^{k-m-1}} \right),$$

we can easily fill in the following table.

<table>
<thead>
<tr>
<th>Field</th>
<th>$f$</th>
<th>$g_1$</th>
<th>$H_1 x_1$ (for $k &gt; 1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$-x_1$</td>
<td>0</td>
<td>$(k - 1)y_k x_1$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$-x_1^2$</td>
<td>0</td>
<td>$(k - 2)y_{k-1} x_1$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$-x_1 y_1$</td>
<td>0</td>
<td>$\sum_{m=2}^{k-1} (m - 1)y_{k-m} x_1$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>$x_1$</td>
<td>$-y_1$</td>
<td>$-ky_k x_1$</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>$x_1 y_1$</td>
<td>$-y_1^2$</td>
<td>$\sum_{m=2}^{k-1} m\lambda_{k-m+1} x_1$</td>
</tr>
</tbody>
</table>

As we have seen, a vector field will vanish at $P_k$ iff the associated $H_1$ does. We note that the $H_1$ terms for $\beta$, $\gamma$, and $\zeta$ do not depend on the choice of $\lambda_k$, and so if these fields vanish for some $P_k$, they must vanish on the entire exceptional divisor. The field $\beta$ vanishes on the divisor if either $\lambda_{k-1} = 0$, or $k \equiv 2 \mod p$. On the other hand, $\alpha$ and $\epsilon$ only vanish for $\lambda_k = 0$, or $k \equiv 1 \mod p$ (for $\alpha$), or $k \equiv 0 \mod p$ (for $\epsilon$). In these cases, $\alpha$ and $\epsilon$ vanish on the entire exceptional divisor.

Suppose $\lambda_1, \ldots, \lambda_n = 0$. Then the fields $\gamma$ and $\zeta$ will vanish on the exceptional divisor for all $k < 2n + 1$. 

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Applying these observations to the example discussed in §2 above, we recall that the only nonzero \( \lambda_k \) occurred for \( k = 2p + 1 \) and \( k = 3p + 1 \). Thus, \( \alpha, \beta, \gamma, \) and \( \xi \) all vanish at \( P_k \) for all \( k < 3p + 1 \), and \( V_{B_{3p+2}} \) has dimension four. But as we saw, \( \text{Aut } B_{3p+2} \) has dimension three. This provides the desired example.

4. Alternatives. The methods used to calculate automorphisms and vector fields, although applied here to a specific example, are general enough to describe many others. The simplest variation would be to postpone the second nonzero \( \lambda \) to some distant step, also congruent to 1 mod \( p \); the same result will follow. Instead of using the fields \( \alpha \) and \( \beta \), we could take the only nonzero \( \lambda \)'s at steps congruent to 0 mod \( p \), and construct an example using the field \( e \).

Finally, we can extend the present example until there are no automorphisms left, and still have vector fields. For example, in characteristic 2, if the only nonzero \( \lambda \)'s are at \( k = 5, 7, 9, 13, \) and \( 17 \), then equation (2.2) shows that the only automorphism fixing \( P_{17} \) is the identity \( (e = i = 1, b = c = f = 0) \), but the vector fields \( \alpha \) and \( \beta \) both lift to \( B_{17} \) and vanish at \( P_{17} \). (Of course, characteristic 2 is a little special here, because the expressions \( (k - 1)\lambda_k \) and \( (k - 2)\lambda_{k-1} \) can only be synchronized in characteristic 2. Normally, we can only arrange for a single vector field (e.g., \( \alpha \)) to lift to \( B_n \) with discrete automorphism group.)

REFERENCES


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