THE SCHUR INDEX OF THE $p$-REGULAR CHARACTERS OF THE BOREL SUBGROUP

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Abstract. Let $B$ be the group of $\mathbb{F}_q$-rational points of the Borel subgroup of a connected reductive group defined over the finite field $\mathbb{F}_q$. It is shown that under appropriate conditions, all irreducible characters of $B$ which have degree prime to $q$ have Schur index one.

Let $G$ be a connected reductive linear algebraic group defined over the finite field $\mathbb{F}_q$, with corresponding Frobenius map $F$. We assume that $G$ has connected centre $Z$ and that the characteristic $p$ is good for $G$. If $H$ is a closed subgroup of $G$, $H$ will denote its group of $\mathbb{F}_q$-rational points; for a finite group $H$, a complex irreducible character is $p$-regular if $p$ does not divide its degree; $m_p(H)$ is the number of such characters. It was shown in [2] that if $B$ is an $F$-stable Borel subgroup of $G$, then $m_p(B) = m_p(G)$, verifying a special case of Alperin's conjecture for finite groups. Ohmori [3] has shown that the $\mathbb{Z}/q^l$ (where $l$ is the semisimple rank of $G$) $p$-regular characters of $G$ all have Schur index one, i.e. can be realized in their field of characters. In this note we prove the corresponding result for $p$-regular characters of $B$, viz:

Theorem. The $p$-regular characters of $B$ all have Schur index one.

In the proof, we use the notation of [1] and [2]. $B$ is the semidirect product $T \ltimes U$, where $U$ is a maximal unipotent subgroup of $G$. Correspondingly, $B = T \ltimes U$, where $U$ is a $p$-group and $T$ an abelian $p'$-group.

Proposition 1. Let $\chi$ be a $p$-regular character of $B$. Then $\chi = (\mu \phi)^B$, where $\mu$ is a linear character of $U$ and $\phi$ is a character of the centralizer $T(\mu)$ of $\mu$ in $T$. (Here $\mu \phi$ is a character of $T(\mu) \ltimes U$).

This is elementary and was noted in [2].

Corollary 1'. (i) $\chi|_{T(\mu) \cdot U} = \varphi \cdot \sum_{\ell \in T/\Pi(\mu)} \mu^\ell$;
(ii) $\chi$ vanishes outside $T(\mu) \cdot U$.

Proof. (i) is a simple application of Frobenius' formula for induced characters, and (ii) follows since $T(\mu) \cdot U$ is normal in $B$.

Proposition 2. $\sum_{\ell \in T/\Pi(\mu)} \mu^\ell$ takes rational values on $U$. 

Received by the editors January 31, 1978.


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0002-9939/79/0000-0401/$01.75
Proof. We have a canonical isomorphism \([1, p. 258]\) \(\eta: U/U' \to X_1 \times \cdots \times X_n\), where \(X_i \cong (F_q^n)^+\). Thus we speak of the support of \(\mu\), defined as \(\{i|\mu(X_i) \neq 1\}\). Because of our assumption of connected centre, \(T\) acts transitively on the set of linear characters of \(U\) with given support (by the argument used to prove Theorem B' in [1]). Hence \(\sum_{t \in T/T(\mu)} \mu'\) is the sum of all linear characters of \(U\) with fixed support \(I\). By applying Galois automorphisms, one sees that this sum always takes rational values.

For any character \(\xi\) of some finite group, denote by \(Q(\xi)\) the algebraic extension of \(Q\) obtained by adjoining all the values of \(\xi\). This is the character field of \(\xi\). From Proposition 2 we have immediately:

**Corollary 2'.** If \(\chi\) is a \(p\)-regular character of \(B\), and \(\phi\) is as in Proposition 1, then \(Q(\chi) = Q(\phi)\).

**Lemma 3.** The restriction of \(\chi\) to \(T\) is \(\phi^T_{T(\mu)}\).

**Proof.** This follows directly from Corollary 1' by evaluation of \(\chi|_T\), or by applying Mackey's subgroup theorem.

**Lemma 4.** \(\phi_{T(\mu)}\) has an extension \(\tilde{\phi}\) to \(T\), such that \(Q(\tilde{\phi}) = Q(\phi)\).

**Proof.** If \(Z\) is trivial then \(T = T_1 \times T(\mu)\), since \(T\) is a direct product of the groups \(F_q^*\) and the condition \(t \in T(\mu)\) is that certain components be trivial. In the general case, since the \(Z\)-span of the fundamental roots has a Frobenius-invariant complement in the character group \(X(T)\), we have \(T \cong Z \times T/Z\); using Lang's theorem it follows that \(T = Z \times T/Z\). Hence \(T(\mu)\) is again a direct factor (containing \(Z\)) and if \(T = T(\mu) \times T_1\), we may take \(\tilde{\phi}(t, 1) = \phi(t)\).

**Corollary 4'.** The multiplicity \((\chi, \tilde{\phi}^T) = 1\).

**Proof.** From Lemmas 3 and 4, \((\chi, \tilde{\phi})_T = (\phi^T_{T(\mu)}, \tilde{\phi}) = 1\). The corollary now follows by Frobenius reciprocity.

**Proof of the Theorem.** Let \(\chi, \phi\) be as in Proposition 1. From Corollary 2' we have \(Q(\chi) = Q(\phi)\). The theorem is therefore proved if \(\chi\) can be realized over \(Q(\phi)\). But (with \(\tilde{\phi}\) as in Lemma 4) \(\phi^B_{\tilde{\phi}}\) is a representation of \(B\) in \(Q(\tilde{\phi}) = Q(\phi)\), which contains a representation whose character is \(\chi\) with multiplicity one. Hence \(\chi\) can be realized over \(Q(\phi)\).

**Corollary.** Let the integers \(n_i\) be as in the proof of Proposition 2, and let \(n = \text{g.c.d.}(n_i)\). Then \(Q((q^n - 1)/\sqrt{1})\) is a splitting field for the \(p\)-regular characters of \(B\).
REFERENCES


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