PERIODIC MODULES WITH LARGE PERIODS

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Abstract. Let G be a nonabelian group of order $p^3$ and exponent $p$, where $p$ is an odd prime. Let $K$ be a field of characteristic $p$. In this paper it is proved that there exist periodic $KG$-modules whose periods are $2p$. Some examples of such modules are constructed.

1. Introduction. Let $G$ be a finite $p$-group, and let $K$ be a field of characteristic $p$, where $p$ is a prime integer. If $M$ is a $KG$-module then there exists a projective $KG$-module $F$ such that there is an epimorphism $\varphi: F \to M$. The kernel of $\varphi$ can be written as $\Omega(M) \oplus E$ where $E$ is projective and $\Omega(M)$ has no projective submodules. It is well known [5] that the isomorphism class of $\Omega(M)$ is independent of the choice of $F$ and $\varphi$. Inductively we define $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ for all integers $n > 1$. A $KG$-module is said to be periodic if there exists an integer $n > 0$ such that $M = \Omega^n(M) \oplus P$ where $P$ is a projective $KG$-module. If $n$ is the smallest such integer then $n$ is called the period of $M$.

Recently it has been proved that, when $G$ is abelian, every periodic $KG$-module has period 1 or 2 (see [2] or [4]). It is well known that if $G$ is a quaternion group, then every $KG$-module has period 1, 2 or 4. Until now there were no known examples of periodic modules with periods other than 1, 2, or 4 (see [1]). In this paper we show that if $G$ is a group of order $p^3$ and exponent $p$ for $p$ an odd prime, then there exist periodic $KG$-modules with period $2p$. Some examples along with their minimal projective resolutions are explicitly constructed.

2. Notation and preliminaries. Let $p$ be an odd prime integer. Suppose that $K$ is a field of characteristic $p$. Throughout the rest of this paper $G$ will denote the group of order $p^3$ and exponent $p$. Then $G$ is generated by two elements $x$ and $y$. If $z = x^{-1}y^{-1}xy$, then we have the relations $x^p = y^p = z^p = 1$, $xz = zx$, and $yz = zy$. Let $H$ be the subgroup generated by $x$ and $z$. Let

$$\tilde{H} = \sum_{h \in H} h = (x - 1)^{p-1}(z - 1)^{p-1} \in KH.$$ 

Recall that a $KH$-module $L$ is free if and only if $\dim_K \tilde{H}L = (1/p^2)\dim_K L$. If $M$ is a $KG$-module, then $M_H$ is its restriction to a $KH$-module.

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The remainder of this section is devoted to establishing some combinatorial relations which will be needed in the next section. For \( \alpha \in K \), let \( l = l(\alpha) = (\gamma - 1) - \alpha(x - 1) \in KG \).

**Lemma 2.1.** \( l^p = k(z - 1)^{p^{-1}} \) where

\[
k = \sum_{i=1}^{p-1} (-1)^i \frac{1}{p} \binom{p}{t} \alpha^t z^{p^{-i}}.
\]

**Proof.** Note that \( l = (\alpha - 1) + (\gamma - ax) \). Therefore

\[
l^p = (\alpha - 1)^p + (\gamma - ax^p) = \alpha^p - 1 + (\gamma - ax)^p.
\]

In the expansion of \((\gamma - ax)^p\) the coefficient on \((-\alpha)^i\) is the sum of all possible products of \(x\)'s and \(y\)'s with \(x\) occurring \(t\) times and \(y\) occurring \(p - t\) times. Each such product can be written in the form \(x^t y^{p-t} z^s\) for some \(s = 0, 1, \ldots, p - 1\). Suppose that in each product the left-most letter (either \(x\) or \(y\)) is moved to the furthest right position. This operation amounts to multiplying the product by \(z^t\). However the entire sum remains unchanged.

Hence for \(t = 1, \ldots, p - 1\), the coefficient on \((-\alpha)^i\) is

\[
\sum_{j=0}^{p-1} \frac{1}{p} \binom{p}{t} x^t y^{p-t} z^j = \frac{1}{p} \binom{p}{t} x^t y^{p-t} (z - 1)^{p^{-1}}.
\]

This proves the lemma.

**Lemma 2.2.** There exists an element \( v \in KG \) such that \( lk - kl = v(z - 1) \).
Moreover if \( \alpha \) is an element of \( K \) which is not in the prime field \( F_p \), then \( v \) is a unit in \( KG \).

**Proof.** Now \( lk - kl = (yk - ky) - \alpha(xk - kx) \). So

\[
lk - kl = \sum_{i=1}^{p-1} (-1)^i \frac{1}{p} \binom{p}{t} \alpha^t z^{p^{-i}}(z^{-t} - 1)
\]

\[
- \sum_{i=1}^{p-1} (-1)^i \frac{1}{p} \binom{p}{t} \alpha^{t+1} x^{t+1} y^{p-t}(1 - z^t).
\]

Note that \( z^{-t} - 1 = z^{p^{-t}} - 1 = (z - 1)(z^{p^{-t}-1} + \cdots + z + 1) \). Hence \( lk - kl = v(z - 1) \) where

\[
v = \alpha x z^{-1} - \alpha^p y z^{-1}
\]

\[
- \sum_{i=1}^{p-2} (-1)^i \frac{1}{p} \binom{p}{t+1} (z^{p^{-i-2}} + \cdots + 1)
\]

\[
- \binom{p}{t} (z^{-t+1} + \cdots + 1) \alpha^{t+1} x^{t+1} y^{p-t}.
\]

Let \( \epsilon : KG \to K \) be the augmentation homomorphism given by \( \epsilon(g) = 1 \) for all \( g \in G \). Then
The lemma follows from the fact that \( v \) is a unit if and only if \( e(p) \neq 0 \).

Now if \( r = 1, \ldots, p - 1 \), then

\[
l'r^r - k'l^r = \sum_{j=0}^{r-1} l^j(k - k'l)r^{r-1-j}
\]

\[
= \sum_{j=0}^{r-1} l^j v l^{r-1-j}(z - 1).
\]

Therefore

\[
(l'r - k'l')(z - 1)^{p-2} = r'l'^{-1}v(z - 1)^{p-1}.
\]

3. The main result. Let \( K \) be a field with odd characteristic \( p \). Let \( G \), \( l = l(\alpha) \), \( k, \nu \) be as in the previous section. Let \( W \) be the left ideal in \( KG \) given as \( W = KGl + KG(z - 1) \). We define \( M = M(\alpha) = KG/W \). Then \( M \) is a cyclic \( KG \)-module generated by \( m = 1 + W \) where \( (z - 1)m = 0 \) and \( (y - 1)m = a(x - 1)m \). The dimension of \( M \) is \( p \), and the restriction of \( M \) to a \( K\langle x \rangle \)-module is isomorphic to \( K\langle x \rangle \).

**Theorem 3.1.** If \( \alpha \not\in F_p \) (the prime field), then \( M(\alpha) \) is a periodic \( KG \)-module with period \( 2p \).

The proof consists of constructing a minimal free resolution for \( M \). Let \( F = KGa \oplus KGb \) be a free module with generators \( a \) and \( b \). For each \( i = 1, \ldots, p - 1 \), let

\[
m(i, 1) = l'i'a - (z - 1)b,
\]

\[
m(i, 2) = k(z - 1)^{p-2}a - l'^{-1}b.
\]

Define \( M_i \) to be the submodule of \( F \) generated by \( m(i, 1) \) and \( m(i, 2) \).

**Lemma 3.2.** \( \text{Dim } M_i > p^3 + p \).

**Proof.** Now \( \tilde{H}m(i, 1) = (y - 1)^{t+i}\tilde{H}a \), and \( \tilde{H}m(i, 2) = -(y - 1)^{p-i}\tilde{H}b \).

Therefore the \( KH \)-module

\[
E = \sum_{j=0}^{p-i-1} KH(y - 1)^{t+j}m(i, 1) \oplus \sum_{j=0}^{i-1} KH(y - 1)^{s+j}m(i, 2)
\]

is a free \( KH \)-submodule of \( (M_i)_H \) whose \( KH \)-socle is the subspace with a basis consisting of the elements \( (y - 1)^{t+i}\tilde{H}a, \, t = i, \ldots, p - 1 \), and \( (y - 1)^{s+i}\tilde{H}b, \, s = p - i, \ldots, p - 1 \). Also \( \text{Dim } E = p^3 \). Let

\[
m(i, 3) = k(z - 1)^{p-2}m(i, 1) - l'i'm(i, 2)
\]

\[
= (kl'^{-1} - l'k)(z - 1)^{p-2}a
\]

\[
= -il'^{-1}v(z - 1)^{p-1}a.
\]
The last equality follows from (2.3). By Lemma 2.2, 
\[(x - 1)^{p-1}v^{-1}m(i, 3) = -i(y - 1)^{i-1}\tilde{H}a \notin E.\]
Let
\[L = KH v^{-1}m(i, 3) = K\langle x\rangle v^{-1}m(i, 3).\]
Then \(L \cap E = 0\) and \(\text{Dim } L = p\). Consequently \(L \oplus E\) is a subspace of \(M_i\) of dimension \(p^3 + p\).

**Lemma 3.3.** \(M_1 \cong \Omega^2(M)\) and \(\text{Dim } M_1 = p^3 + p\).

**Proof.** From the definition we know that \(W \cong \Omega(M)\). We have an exact sequence
\[0 \rightarrow \Omega^2(M) \rightarrow F \rightarrow W \rightarrow 0\]
where \(\varphi\) is defined by \(\varphi(a) = z - 1\) and \(\varphi(b) = l\). Then
\[\varphi(m(1, 1)) = l(z - 1) - (z - 1)l = 0.\]
Also \(\varphi(m(1, 2)) = k(z - 1)^{p-1} - l^p = 0\), by Lemma 2.1. Hence \(M_i\) is in the kernel of \(\varphi\). Since \(\text{Dim } W = p^3 - p\), the dimension of the kernel of \(\varphi\) is \(p^3 + p\). By Lemma 3.2, \(M_1\) is the kernel of \(\varphi\).

**Lemma 3.4.** For each \(i = 1, \ldots, p - 2\), \(\Omega^2(M_i) \cong M_{i+1}\). Moreover \(\text{Dim } M_i = p^3 + p\) for all \(i = 1, \ldots, p - 1\).

**Proof.** Assume, by induction, that \(\text{Dim } M_i = p^3 + p\). Note that
\[(z - 1)^{p-1}m(i, 1) - (z - 1)m(i, 2) = 0. \quad (3.5)\]
We also have that
\[lk(z - 1)^{p-2}m(i, 1) - l^{i+1}m(i, 2) = -l(lk - kl')(z - 1)^{p-2}a = -iv(z - 1)^{p-1}m(i, 1).\]
Therefore
\[
[lk + iv(z - 1)](z - 1)^{p-2}m(i, 1) - l^{i+1}m(i, 2) = 0. \quad (3.6)
\]
Let \(F' = KG_c \oplus KG_d\) be the free \(KG\)-module with generators \(c\) and \(d\). We can form the exact sequence
\[0 \rightarrow \Omega(M_i) \rightarrow F' \rightarrow M_i \rightarrow 0,
\]
where \(\psi(c) = m(i, 1)\) and \(\psi(d) = m(i, 2)\). By (3.5) and (3.6), the kernel of \(\psi\) contains the elements

\[u_1 = l^{p-i}c - (z - 1)d\]

and
\[u_2 = [lk + iv(z - 1)](z - 1)^{p-2}c - l^{i+1}d.\]
Now \(\tilde{H}u_1 = (y - 1)^{p-i}\tilde{H}c, \tilde{H}u_2 = (y - 1)^{i+1}\tilde{H}d\). Let
\[u_3 = k(z - 1)^{p-1}c - l'(z - 1)d = l'u_1.\]
Clearly \((z - 1)^{p-1}u_3 = 0\) and
\[(x - 1)^{p-1}(z - 1)^{p-2}u_3 = -(y - 1)^{i}\tilde{d}.
\]
By an argument similar to that in Lemma 3.2, we get that the dimension of the module \(L\), generated by \(u_1\) and \(u_2\), is at least \(p^3 - p\). Since the dimension of the kernel of \(\psi\) is \(p^3 - p\), \(L \cong \Omega(M_i)\).

We can form the exact sequence
\[0 \to \Omega^2(M_i) \to F^\theta \to L \to 0\]
where \(\theta(a) = u_1\) and \(\theta(b) = u_2\). It is easy to see that
\[\theta(m(i + 1, 1)) = i^{i+1}u_1 - (z - 1)u_2 = 0.
\]
Also
\[\theta(m(i + 1, 2)) = k(z - 1)^{p-2}u_1 - l^{p-i}u_2
= [kp^{i-1} - l^{p-i}k - ivp^{i-1}(z - 1)](z - iy-2c = 0,
by (2.3). Consequently \(M_{i+1}\) is in the kernel of \(\theta\). By Lemma 3.2, \(M_{i+1}\) is the kernel of \(\theta\).

To conclude the proof of Theorem 3.1 we need only the following.

**Lemma 3.7.** \(\Omega^2(M_{p-1}) \cong M\).

**Proof.** We have an exact sequence
\[0 \to \Omega(M_{p-1}) \to F' \to M_{p-1} \to 0\]
where \(F' = KGc \oplus KGd\), \(\sigma(c) = m(p - 1, 1)\) and \(\sigma(d) = m(p - 1, 2)\). Let \(u = lc - (z - 1)d\). Then \(\sigma(u) = 0\). Now \(\tilde{H}u = (y - 1)\tilde{H}c\) and
\[(x - 1)^{p-1}(z - 1)^{p-2}l^{p-1}u = -(y - 1)^{p-1}\tilde{H}d \neq 0.
\]
By an argument similar to that of Lemma 3.2, we get that \(\dim KGu \geq p^3 - p\). Therefore the kernel of \(\sigma\) is \(KGu \cong \Omega(M_{p-1})\).

Define \(\tau: KG \to KGu\) by \(\tau(1) = u\). The kernel of \(\tau\) has dimension \(p\) and is isomorphic to \(\Omega^2(M_{p-1})\). Let \(\omega = (z - 1)^{p-1}l^{p-1}\). Then \(\tau(\omega) = 0\) and \((z - 1)\omega = l\omega = 0\). Since \((x - 1)^{p-1}\omega = (y - 1)^{p-1}\tilde{H} \neq 0\), \(KG\omega\) is the kernel of \(\tau\), and \(M = KG\omega\). This completes the proof of the lemma and the theorem.

It should be noted that if \(\alpha \in F_p\) then \(M(\alpha)\) is not periodic. This follows from the fact that the restriction of \(M(\alpha)\) to the subgroup \(J = \langle x^{-1}y, z \rangle\) is not a periodic module (see [2]). It remains to show that there exist periodic modules with period \(2p\) when \(K = F_p\).

Let \(f = T^n + \beta_{n-1}T^{n-1} + \cdots + \beta_1T + \beta_0\) be an irreducible polynomial in \(K[T]\). Let \(L\) be the \(KG\)-module of dimension \(np\) on which \(x\) and \(y\) are represented by the matrices
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respectively, where $I$ is the $n \times n$ identity matrix, and

$$
A_x = \begin{bmatrix}
I & & \\
I & I & \\
& \ddots & \ddots \\
& & I & I
\end{bmatrix},
\quad A_y = \begin{bmatrix}
I & & \\
U & I & \\
& \ddots & \ddots \\
& & U & I
\end{bmatrix},
$$

is the companion matrix for $f$. Now $L$ is an indecomposable $KG$-module. If $K'$ is an extension of $K$ which splits $f$, then

$$
K' \otimes_K L \cong M(\alpha_1) \oplus \cdots \oplus M(\alpha_n)
$$

where $\alpha_1, \ldots, \alpha_n$ are the roots of $f$ in $K'$. If therefore $n > 1$, then $L$ is periodic with period $2p$ since, by the Noether-Deuring Theorem (see [3, 29.7]),

$$
K' \otimes \Omega^{2p}(L) \cong \Omega^{2p}(K' \otimes L) \cong K' \otimes L
$$

implies that $\Omega^{2p}(L) \cong L$.

The reader is invited to check that $L$ is periodic with period $2p$ when the matrix $U$ is replaced by

$$
U' = \begin{bmatrix}
\alpha \\
1 & \alpha \\
& \ddots & \ddots \\
& & 1 & \alpha
\end{bmatrix}
$$

for $\alpha \in K$, $\alpha \notin F_p$. Combining this with the fact that $M(\alpha) \cong M(\beta)$ if and only if $\alpha = \beta$, we get the following.

**Theorem 3.8.** Let $p$ be an odd prime and let $K$ be a field of characteristic $p$. If $G$ is the nonabelian group of order $p^3$ and exponent $p$, then there exist periodic $KG$-modules which have period $2p$. Moreover there exist an infinite number of isomorphism classes of such modules and there exist such modules with arbitrarily large dimension.
REFERENCES


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