THE STRONGLY PRIME RADICAL ¹

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Abstract. Let $R$ denote a strongly prime ring. An explicit construction is given of the radical in $R$-mod corresponding to the unique maximal proper torsion theory. This radical is characterized in two other ways analogous to known descriptions of the prime radical in rings. If $R$ is a left Ore domain the radical of a module coincides with the torsion submodule.

1. The strongly prime radical. The terminology of radicals in modules is that of Stenstrom [3]. Throughout this paper all rings have a unity and all modules are unital left modules. For a ring $R$ the category of $R$-modules is denoted by $R$-mod.

A functor $\sigma: R$-mod $\rightarrow R$-mod is called a preradical if $\sigma(M)$ is a submodule of $M$ and $\sigma(M) \alpha \subseteq \sigma(N)$ for each morphism $M \xrightarrow{\alpha} N$ in $R$-mod. A preradical $\alpha$ is called a radical if $\sigma(M/\sigma(M)) = 0$ for all $M \in R$-mod. A preradical $\sigma$ is called left exact if $\sigma(N) = N \cap \sigma(M)$ whenever $N \subseteq M$ in $R$-mod (equivalently, if $\sigma$ is a left exact functor). One method of constructing left exact radicals is given by the following result.

Proposition 1. Let \( \mathcal{M} \) be any nonempty class of modules closed under isomorphisms. For any module $M$ define
\[
\sigma(M) = \bigcap \{ K \mid K \subseteq M, M/K \in \mathcal{M} \}.
\]
It is assumed that $\sigma(M) = M$ if $M/K \notin \mathcal{M}$ for all $K \subseteq M$. Then
(1) $\sigma(M/\sigma(M)) = 0$ for all modules $M$;
(2) if $\mathcal{M}$ is closed under taking nonzero submodules, $\sigma$ is a radical;
(3) if $\mathcal{M}$ is closed under taking essential extensions, then $\sigma(M) \cap N \subseteq \sigma(N)$ for all submodules $N \subseteq M$.

In particular, $\sigma$ is a left exact radical if $\mathcal{M}$ is closed under nonzero submodules and essential extensions.

Proof. The proofs of (1) and (2) are straightforward and so are omitted; the last sentence follows from (2) and (3). To prove (3) let $N \subseteq M$ be modules. We must verify that $N \cap \sigma(M) \subseteq K$ whenever $N/K \in \mathcal{M}$. By Zorn's lemma, choose $W$ maximal in
\[
\mathcal{S} = \{ W \mid K \subseteq W \subseteq M, W \cap N = K \}.
\]
We claim that \((W + N)/W\) is essential in \(M/W\). For if
\[
X/W \cap (W + N)/W = 0
\]
where \(X/W \neq 0\) then \(X \supset W\) so \(X \cap N \supset K\) by the choice of \(W\). Suppose
\(x \in (X \cap N) - K\). Then
\[
x + W \in X/W \cap (W + N)/W = 0
\]
so \(x \in N \cap W = K\), a contradiction. Hence \((W + N)/W \subseteq M/W\) is essential. Since
\[
(W + N)/W = N/(W \cap N) = N/K \subseteq \mathcal{M},
\]
it follows that \(M/W \in \mathcal{M}\) and so \(\sigma(M) \subseteq W\). Thus \(\sigma(M) \cap N \subseteq W \cap N = K\) as required. □

We are going to apply this to the following class of modules: An \(R\)-module
\(M\) is called strongly prime \([1]\) if \(M \neq 0\) and, for each nonzero element
\(m \in M\), there exists a finite subset \(\{r_1, \ldots, r_k\} \subseteq R\) (depending on \(m\)) such
that \(r_i m = 0\) for all \(i (r \in R)\) implies \(r = 0\). In \([1]\) the set \(\{r_1, r_2, \ldots, r_k\}\)
is called an insulator for \(m\). A ring \(R\) is called left strongly prime if \(R\) is
strongly prime (this is not left-right symmetric \([1, p. 212]\)).

Proposition 2. The class of strongly prime modules is closed under taking
isomorphic images, (nonzero) submodules and essential extensions.

Proof. It is obviously closed under isomorphic images and nonzero sub-
modules. If \(M \subseteq X\) is an essential extension and \(M\) is strongly prime let
\(0 \neq x \in X\). Then \(Rx \cap M \neq 0\), say \(0 \neq rx \in M, \ r \in R\). Then if
\(\{r_1, \ldots, r_k\}\) is an insulator for \(rx\) it is clear that \(\{rxr, \ldots, rkr\}\) is an insulator
for \(x\). □

Now define the strongly prime radical \(\beta\) on \(\mathbb{R}\)-mod by
\[
\beta(M) = \bigcap \{K | K \subseteq M, M/K\text{ strongly prime}\},
\]
where we assume that \(\beta(M) = M\) whenever \(M\) has no strongly prime images. Observe that every strongly prime module is faithful. If \(M\) is strongly prime and
\(0 \neq r \in R\) then \(rM \neq 0\), say \(rm \neq 0, \ m \in M\). If \(\{r_1, r_2, \ldots, r_k\}\) is an
insulator for \(rm\) then it is also an insulator for \(r\) in \(R\). It follows that \(R\) is left
strongly prime if and only if it has a strongly prime module \([1, p. 220]\). In
particular, \(\beta(M) = M\) for all \(M \in \mathbb{R}\)-mod unless \(R\) is a strongly prime ring. In
this case Propositions 1 and 2 give:

Proposition 3. If \(R\) is left strongly prime then \(\beta\) is a left exact preradical on
\(\mathbb{R}\)-mod.

If \(\sigma\) and \(\rho\) are two preradicals on \(\mathbb{R}\)-mod, we say that \(\rho\) is larger than \(\sigma\)
(written \(\rho > \sigma\)) if \(\rho(M) \supseteq \sigma(M)\) for every module \(M \in \mathbb{R}\)-mod. Then we
have:

Theorem 1. Let \(R\) be left strongly prime. Then \(\beta(R) = 0\) and \(\beta \geq \sigma\) for
every left exact preradical \(\sigma\) on \(\mathbb{R}\)-mod such that \(\sigma(R) = 0\).
Proof. Clearly $\beta(R) = 0$ (since $\_R$ is strongly prime). Suppose $\sigma$ is a preradical on $R$-mod for which $\sigma(R) = 0$. Given $M \in R$-mod we must show $\sigma(M) \subseteq \beta(M)$. If not then $\sigma(M) \not\subseteq K$ for some $K \subseteq M$ with $M/K$ strongly prime. If $\alpha: M \to M/K$ is the natural map then 

$$0 \neq \left[\sigma(M) + K\right]/K = \sigma(M)\alpha \subseteq \sigma(M/K)$$

so it suffices to show that $\sigma(M) = 0$ whenever $M$ is strongly prime. Suppose on the contrary that $0 \neq m \in \sigma(M)$. Let $\{r_1, \ldots, r_k\}$ be an insulator for $m$ and define $\lambda: R \to M^k$ by $r\lambda = (rr_1m, \ldots, rr_km)$. This is an $R$-monomorphism and $R\lambda \subseteq \sigma(M)^k \subseteq \sigma(M^k)$. But $\sigma$ is left exact so $\sigma(R) = 0$ implies 

$$0 = \sigma(R\lambda) = R\lambda \cap \sigma(M^k) = R\lambda,$$

a contradiction. □

A nonempty class $\mathfrak{T}$ of modules is called a pretorsion class if it is closed under quotients and direct sums; if in addition $\mathfrak{T}$ has the property that $M/K$, $K \in \mathfrak{T}$ imply $M \in \mathfrak{T}$, then $\mathfrak{T}$ is called a torsion class. A pretorsion class is called hereditary if it is closed under taking submodules. If a preradical $\sigma$ on $R$-mod is given, the class $\mathfrak{T}_\sigma = \{M | \sigma(M) = 0\}$ is known to be a pretorsion class and the assignment $\sigma \leftrightarrow \mathfrak{T}_\sigma$ is a bijection between left exact preradicals and hereditary pretorsion classes [3, p. 138] under which left exact radicals correspond with hereditary torsion classes [3, p. 139]. In particular, if $R$ is left strongly prime and $\beta$ is the strongly prime radical on $R$-mod, then Theorem 1 implies that $\mathfrak{T}_\beta$ is the torsion class of the largest hereditary torsion theory [3, p. 141] on $R$-mod for which $R$ is torsion-free. The existence of a unique maximal proper torsion theory on $R$-mod was given in [1, p. 220].

2. Further characterizations of the strongly prime radical. In this section we present two characterizations of the strongly prime radical which are analogs of well-known descriptions of the prime radical of a ring. The first gives a generalization of the notion of an $m$-system. A subset $X$ of an $R$-module $M$ is called an $fm$-system if $X \neq \emptyset$ and for each $x \in X$ there is a finite subset $F \subseteq R$ (depending on $x$) such that $rFx \cap X \neq \emptyset$ for all $0 \neq r \in R$.

Proposition 4. If $N \subseteq M$ are modules then $M/N$ is strongly prime if and only if $M - N$ is an $fm$-system.

Proof. If $M - N$ is an $fm$-system then $M/N \neq 0$ and, if $m \notin N$ for some $m \in M$, choose $F = \{r_1, \ldots, r_k\} \subseteq R$ such that $rFm \cap (M - N) \neq \emptyset$ for all $0 \neq r \in R$. Then $F$ is an insulator for $m + N$. For the converse, reverse the argument. □

One immediate consequence of this proposition is that subdirect products of strongly prime modules are strongly prime. Alternatively, if $K_i \subseteq M$, $i \in I$, are submodules such that $M/K_i$ is strongly prime for each $i \in I$, then $M/\cap K_i$ is strongly prime. This follows since $M - \cap K_i = \cup (M - K_i)$ and the union of a collection of $fm$-systems is again an $fm$-system. In
particular, either $\beta(M) = M$ or $M/\beta(M)$ is strongly prime for every module $M$.

**Theorem 2.** Let $R$ be a left strongly prime ring and let $\beta$ denote the strongly prime radical in $R$-mod. Then

$$\beta(M) = \{m \in M\mid each \text{fm-system } X \text{ with } m \in X \text{ has } 0 \in X\}$$

holds for each module $M$. Furthermore $\beta(M)$ is the unique smallest submodule of $M$ with the property that $M/\beta(M)$ is strongly prime or zero.

**Proof.** The last sentence follows by the preceding remark. Write

$$B = \{m \in M\mid each \text{fm-system } X \text{ with } m \in X \text{ has } 0 \in X\}.$$ 

If $\beta(M) \neq M$ then $M - \beta(M)$ is an fm-system which does not contain zero so $B \subseteq \beta(M)$ in this case. This clearly holds if $\beta(M) = M$.

Now suppose $m \notin B$; we must show $m \notin \beta(M)$. There is an fm-system $X$ with $m \in X$ and $0 \notin X$. Let $S = \{K \subseteq M\mid K \text{ a submodule and } K \cap X = \emptyset\}$. Then $0 \in S$ and, by Zorn's lemma, we may choose a maximal member $K$ of $S$. Since $m \notin K$ we are finished if we can show that $M/K$ is strongly prime, equivalently that $M - K$ is an fm-system. Given $m_1 \in M - K$ then $Rm_1 + K$ meets $X$ by the maximality of $K$, say $rm_1 + k = x \in X$. Since $X$ is an fm-system, choose a finite set $F = \{r_1, \ldots, r_t\} \subseteq R$ such that $sFx \cap X \neq \emptyset$ for each $0 \neq s \in R$. If $sr_ix \in X$ for such an $s$, then $srirm_1 + sr_ik \in X$. But $K \cap X = \emptyset$ and $sr_ik \in K$ so it follows that $srirm_1 \notin K$. Thus

$$srirm_1 \in s(Fr)m_1 \cap (M - K)$$

and so $M - K$ is an fm-system as required. □

**Note** that this argument yields slightly more. If $X_0$ is any fm-system with $0 \notin X_0$ then Zorn's lemma produces a maximal fm-system $X \supseteq X_0$ with $0 \notin X$. If we now choose $K$ as in the proof of Theorem 2 then $X \subseteq M - K$ (since $X \cap K = \emptyset$) and hence $X = M - K$ by the maximality of $X$. Thus

**Corollary.** If $X$ is a maximal fm-system such that $0 \notin X$ in a module $M$ then $K = M - X$ is a submodule with $M/K$ strongly prime. In particular, a module $M$ contains an fm-system $X$ with $0 \notin X$ if and only if $M$ has a strongly prime image.

We now turn to a characterization of the strongly prime radical which is analogous to the lower radical construction of the prime radical of a ring. Given a module $M$, inductively define an ascending chain of submodules $M_\lambda$, $\lambda$ an ordinal, as follows:

1. $M_0 = 0$;
2. if $\lambda$ is a limit ordinal, define $M_\lambda = \bigcup_{\mu < \lambda} M_\mu$;
3. if $\lambda = \mu + 1$, define

$$M_\lambda = M_{\mu + 1} = \left\{m \in M \mid \text{given a finite nonempty subset } F \subseteq R, \text{ there exists } 0 \neq r \in R \text{ such that } rFm \subseteq M_\mu\right\}.$$
Clearly \( M_\gamma \subset M_{\gamma+1} \) so these \( M_\gamma \) are an ascending chain of submodules. If \( \gamma \) is the least ordinal for which \( M_\gamma = M_{\gamma+1} \) write \( M_\gamma = L(M) \).

**Theorem 3.** If \( R \) is left strongly prime and \( \beta \) is the strongly prime radical in \( R\)-mod then \( \beta(M) = L(M) \) holds for every \( M \in R\)-mod.

**Proof.** Let \( L(M) = M_\gamma = M_{\gamma+1} \). We show first that \( \beta(M) \subset M_\gamma \). If \( M_\gamma = M \) this is clear. Otherwise it suffices to show \( M/M_\gamma \) is strongly prime. If \( m \in M - M_\gamma \) then \( m \in M_{\gamma+1} \) so there exists a finite set \( F = \{ r_1, \ldots, r_k \} \subset R \) such that \( rFm \subseteq M \) for every \( 0 \neq r \in R \). Thus \( rFm \cap (M - M_\gamma) \neq \emptyset \) for all \( 0 \neq r \in R \) so \( M - M_\gamma \) is an \( \ell \)-system as required.

To prove \( M_\gamma \subset \beta(M) \) we prove inductively that \( M_\lambda \subset \beta(M) \) holds for every ordinal \( \lambda \). The only case where proof is required is when \( \lambda = \mu + 1 \) for some ordinal \( \mu \). Assume \( M_\mu \subset \beta(M) \) and suppose \( m \in M_\lambda - M_\mu \). Then, since \( M - \beta(M) \) is an \( \ell \)-system, there exists a finite set \( F \subset R \) with \( rFm \cap (M - \beta(M)) \neq \emptyset \) for all \( 0 \neq r \in R \). But \( m \in M_\lambda = M_{\mu+1} \) means there exists \( 0 \neq r_0 \in R \) such that \( r_0Fm \subseteq M_\mu \). This contradiction shows that \( M_\lambda \subset \beta(M) \) and so completes the induction. \( \square \)

One important class of strongly prime rings is the class of domains. We now relate \( \beta(M) \) for \( M \in R\)-mod to the set of torsion elements \( \tau(M) \), where \( R \) is a domain. Recall that \( \tau(M) = \{ m \in M | rm = 0 \text{ for some } 0 \neq r \in R \} \).

**Proposition 5.** If \( R \) is a domain then \( \beta(M) \subset \tau(M) \) for all \( M \in R\)-mod.

**Proof.** We use Theorem 3 and show inductively that \( M_\lambda \subset \tau(M) \) for every ordinal \( \lambda \). Again we need only discuss the case when \( \lambda = \mu + 1 \) for some ordinal \( \mu \) and \( M_\mu \subset \tau(M) \). Let \( m \in M_\lambda \). Then the definition of \( M_{\mu+1} \) (with \( \{ 1 \} = F \)) shows that there exists \( 0 \neq s \in R \) with \( rm \in M_\mu \). Thus \( srm = 0 \) for some \( 0 \neq s \in R \) and, since \( R \) is a domain, this shows that \( m \in \tau(M) \). \( \square \)

In the case of left Ore domains, Levy [2] has shown that, for each \( M \in R\)-mod, \( \tau(M) \) is a submodule of \( M \). In this case it is easy to verify that \( \tau \) is a left exact radical on \( R\)-mod and it is clear that \( \tau(R) = 0 \). Hence, by Theorem 1, \( \beta \geq \tau \). With Proposition 5 this gives:

**Proposition 6.** If \( R \) is a left Ore domain, then \( \tau(M) = \beta(M) \) for all \( M \in R\)-mod.

3. **The faithful prime radical.** The preceding work can be repeated to deal with the radical determined by the class \( \mathcal{P}_0 \) of faithful prime modules in \( R\)-mod (so we assume \( R \) is a prime ring). Then Proposition 2 is valid for \( \mathcal{P}_0 \) and yields a left exact radical

\[
\beta_0(M) = \{ K | K \subset M, M/K \text{ is faithful and prime} \}
\]

when we set \( \beta_0(M) = M \) if \( M \) has no faithful, prime images. We call \( \beta_0(M) \) the faithful prime radical of \( M \). Clearly \( \beta_0 \leq \beta \) over a strongly prime ring.

Define an \( \ell \)-system in a module \( M \) to be a nonempty subset \( X \) of \( M \) such that, for each \( x \in X \) and \( 0 \neq r \in R, rRx \cap X \neq \emptyset \). Then \( M/N \) is a faithful
prime module if and only if \( M - N \) is an \( m \)-system. Furthermore Theorem 2 has its analog for prime rings \( R \) obtained by replacing \( \beta \), "strongly prime" and "\( fm \)-system" by \( \beta_0 \), "faithful prime" and "\( m \)-system" throughout. The proof is analogous to the above and is omitted.

We also have a lower radical construction of \( \beta_0(M) \). A sequence \( M_\lambda, \lambda \) an ordinal, of submodules of a module \( M \) is defined as before except that, when \( \lambda = \mu + 1 \), we define \( M_{\mu + 1} = \{ m \in M \mid \text{there is an ideal } I \neq 0 \text{ of } R \text{ with } Im \subseteq M_\mu \} \). Again we find that the terminal module in this ascending chain is \( \beta_0(M) \).

Finally, let \( F \) be a field with a monomorphism \( \alpha: F \to F \) which is not onto and let \( R = F[x, \alpha] \) be the skew polynomial ring with coefficients written on the left. Then \( R \) is a left Ore domain which is left primitive. In fact, if \( b \in F - Fa \), then \( M = R/R(x + b) \) is a faithful irreducible module which is torsion. Hence \( \beta_0(M) = 0 \) while \( \tau(M) = \beta(M) = M \) and so \( \beta_0 < \beta \).

References