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STABLE RANGE IN AW^* ALGEBRAS

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ABSTRACT. We show all finite AW^* algebras (and somewhat more general C^* algebras) satisfy a stronger property than having 1 in their stable range.

I. A ring R has 1 *in its stable range* if for all pairs a,b of elements of R, $aR + bR = R$ implies there is an element c in R such that $a + bc$ is invertible (e.g. [1, p. 231]). We show finite AW^* algebras (and somewhat more general C^* algebras) satisfy an even stronger property, that the c may be chosen to be unitary. Some related properties are also studied.

LEMMA 1. *Let $s_1, s_2, \ldots, s_m$ be positive elements of a C^* algebra R. If there is a projection p in R such that $\sum_{i=1}^m s_i R = pR$, then $(\sum s_i)R = pR$. In particular, if $p = 1$, $\sum s_i$ is invertible in R.*

PROOF. Since $ps_i = s_i$, each s_i commutes with p, and since $p = \sum s_i a_i = \sum (ps_ip)a_ip = \sum (ps_ip)(pa_ip)$ for some collection of elements $\{a_i\}$, by reducing to the ring pRp, we may assume $p = 1$. It now suffices to show $s = \sum s_i$ is invertible, and we may assume $s < 1$.

Write $1 = \sum s_i a_i$. Then

$$1 = \left(\sum s_i a_i\right)\left(\sum s_i a_i\right)^* = \sum s_i a_i a_i^* s_i + \sum_{i \neq j} s_i a_i a_j^* s_j.$$

From $3^{m-2} \cdot \sum^m z_i z_i^* \geq \sum_{i \neq j} z_i z_j^*$ (proved by induction on m: consider xx^*, where $x = (\sum^{m-1} z_i) - z_m$; by varying the x, 3^{m-2} may be improved to $2^{[\log_2(m-1)]+1} - 1$), we obtain

$$1 < 3^{m-2} \cdot \left(\sum s_i a_i a_i^* s_i\right).$$

Since $a_i a_i^* < \|a_i\|^2$, $s_i a_i a_i^* s_i < \|a_i\|^2 s_i^2$, so that with $K = \max\{\|a_i\|^2\}$, $1 < 3^{m-2} \cdot K \sum s_i^2$.

From $s_i < 1$, $s_i \geq s_i^2$, and thus $1 < 3^{m-2} \cdot K \sum s_i$, whence $\sum s_i$ is invertible. □

(The preceding proof is an elaboration of the referee's quick proof for the case $m = 2$, and is simpler than my originally intended proof.)

LEMMA 2 (A SLIGHT GENERALIZATION OF [7, p. 122]). *If R is a finite AW^* algebra, then for all a in R, there exists a unitary u such that au is positive.*

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PROOF. Pick a in R, and polar decompose it, i.e. write $a = v(a^*a)^{1/2}$, where v is a partial isometry linking p, the left projection of a, with q, the right projection ([2, p. 133, 1(i)]). As $p \overset{*}{\sim} q$, by [2, p. 106], $1 - p \overset{*}{\sim} 1 - q$; let w be a partial isometry implementing this equivalence ($w^*w = 1 - p$, $w^*w = 1 - q$). Then $u^* = v + w^*$ is a unitary, and as $(1 - p)a = 0$, $w^*a = 0$, so that $u^*a = (a^*a)^{1/2}$; replacing a by a^*, we obtain the desired result. □

Lemma 2 is also true if R is a finite Rickart C^* algebra that is either an $n \times n$ matrix ring (for some n greater than 1) or is monotone σ-complete.

We shall say a C^* algebra satisfies *unitary decomposition* if it satisfies the conclusion of Lemma 2. If au is positive, then $au = (aa^*)^{1/2}$ (for if $au = t \geqslant 0$, $a = tu^*$, so $aa^* = tu^*ut = t^2$).

THEOREM 3. *A C^* algebra satisfying unitary decomposition satisfies unitary 1-stable range: If $aR + bR = R$, there exists a unitary u such that $a + bu$ is invertible. In particular, finite AW^* algebras satisfy unitary 1-stable range.*

PROOF. There exist unitaries u_1, u_2 so that au_1 and bu_1u_2 are positive. Then $au_1R + bu_1u_2R = R$, so by Lemma 1, $au_1 + bu_1u_2$ is invertible. Thus $a + bu_1u_2u_1^* = (au_1 + bu_1u_2)u_1^*$ is invertible in R. The final sentence is a consequence of Lemma 2. □

At this point, we can obtain that for finite type AW^* algebras the algebraic K_1 is simply the abelianized group of units. The type I part is a little tedious to deal with, and we prepare the way with a few lemmas.

LEMMA 4. *Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a collection of positive real numbers whose product is 1. There exists a rearrangement of $\{\lambda_i\}$, call it $\{\theta_i\}$, so that*

(i) $|\log \theta_1| = \max|\log \lambda_i|$,

(ii) $|\log \theta_2| = \max\{|\log \lambda_i| \ \ |\operatorname{sgn} \log \lambda_i = -\operatorname{sgn} \log \theta_1\}$, *and* $\operatorname{sgn} \log \theta_2 = -\operatorname{sgn} \log \theta_1$,

(iii) (a) *if $\theta_1 \geqslant \theta_2$, then for all $k = 1, 2, \ldots, n$, $\theta_1 \geqslant \prod_{i \leqslant k} \theta_i \geqslant \theta_2$,*

(b) *if $\theta_1 < \theta_2$, then for all $k = 1, 2, \ldots, n$, $\theta_2 \geqslant \prod_{i \leqslant k} \theta_i \geqslant \theta_1$.*

PROOF. Select θ_1 from $\{\lambda_i\}$ so that it satisfies (i). If $\theta_1 = 1$, then all the λ_i are 1, so no rearrangement is necessary. Otherwise we may select θ_2 from the remaining λ_i, so that (ii) is satisfied. By replacing $\{\lambda_i\}$ by $\{\lambda_i^{-1}\}$ if necessary, we may assume $\theta_1 > \theta_2$, so $\theta_1 > 1 > \theta_2$, and thus $\theta_1 > \theta_1\theta_2 > \theta_2$.

If $\{\theta_1, \theta_2, \ldots, \theta_t\}$ have been chosen so that (iii) holds for all k less than or equal t, then we define θ_{t+1} by picking it from the remaining λ_i so that is satisfies:

(A) If $\prod^t \theta_i = 1$, then

$$|\log \theta_{t+1}| = \max\{|\log \lambda_j| \ |\lambda_j \in \{\lambda_i\} - \{\theta_1, \theta_2, \ldots, \theta_t\}\}.$$

(B) If $\prod^t \theta_i \neq 1$, then

$$|\log \theta_{t+1}| = \max\{|\log \lambda_j| \ |\lambda_j \in \{\lambda_i\} - \{\theta_i\} i < t, \operatorname{sgn} \log \lambda_j = -\operatorname{sgn} \log(\pi^t \theta_i)$$

and

$$\operatorname{sgn}\log\theta_{t+1} = -\operatorname{sgn}\log\left(\prod^{t}\theta_i\right).$$

It suffices to show $\theta_1 \geqslant \prod^{t+1}\theta_i \geqslant \theta_2$. If $\prod^t \theta_i < 1$, then $\theta_{t+1} > 1$, so $\prod^{t+1}\theta_i > \theta_2$ (by (iii), with $k = t$); as $\theta_1 \geqslant \theta_{t+1}$, and the product of the first t θ_i is less than 1, we also see that $\theta_1 \geqslant \prod^{t+1}\theta_i$. If instead, $\prod^t \theta_i > 1$, then $\theta_{t+1} < 1$, and the dual argument works. Finally, if the product of the first t θ_i's is 1, then conditions (i) and (ii) guarantee the inequalities hold. □

LEMMA 5. *Let X be an extremely disconnected space, and set $R = M_n C(X)$, the ring of $n \times n$ matrices over $C(X)$.*

(a) *If u is a unitary in R, there exist unitaries v, w in R, and f in the centre of R, so that $u = fvwv^{-1}w^{-1}$.*

(b) *If y is a positive invertible element of R, and θ any positive real number, there exist a unitary v in R, a positive invertible z in R, and a central positive element f, so that*

$$y = fzvz^{-1}v^{-1}, \qquad f(x) = (\det y(x))^{1/n},$$

and

$$\max\{\|z\|, \|z^{-1}\|\} \leqslant \max\{\|y\|, \|\det y^{-1}\|^{1/n}, \|y^{-1}\|\cdot\|\det y\|^{1/n}\} + \theta.$$

REMARK. The proof of (a), and a portion of the proof of (b) is adapted from a small part of the proof of the main result of [5].

PROOF. (a) Suppose to begin with that $(\det u)(x) = 1$ for all x in X. One can show that since u is normal and X is sufficiently disconnected, there exists a set of orthogonal projections $\{p_i\}_{i=1}^n$, with $\sum p_i = 1$, $p_j \overset{*}{\sim} p_i$ for all i, j, so that $up_i = t_i p_i$, where t_i: $X \to T$. It is straightforward to show u is unitarily equivalent to the diagonal matrix, $\operatorname{diag}(t_1, t_2, \ldots, t_n)$, with t_i in $C(X, T)$, and $\prod t_i(x) = 1$ for all x.

Define the matrices

$$w = \operatorname{diag}(1, t_1, t_1t_2, t_1t_2t_3, \ldots, t_1t_2\cdots t_{n-1})$$

and

$$v = \begin{pmatrix} 0 & 0 & & & \cdots & 0 & 1 \\ 1 & 0 & & & & & 0 \\ 0 & 1 & 0 & & \mathbf{0} & & \vdots \\ & & \ddots & \ddots & & & \vdots \\ & \mathbf{0} & & & \ddots & 0 & \\ & & & & & 1 & 0 \end{pmatrix}.$$

One checks immediately that $wv = (\operatorname{diag}(t_i^{-1})vw)$; since u is unitarily equivalent to $\operatorname{diag}(t_i)$, u is a unitary commutator.

Now let u be an arbitrary unitary. The function $\det u$: $X \to T$ is a unitary in the centre of R. Since X is extremely disconnected, there exists a positive element $0 \leqslant t \leqslant 1$ in $C(X)$, so that $\det u = \exp(2\pi i t)$. Define $f = \exp(-2\pi i t/n)$. Then $(\det fu)(x) = 1$ for all x, so the previous paragraph applies to fu.

(b) Assume that $(\det y)(x) = 1$ for all x in X. Diagonalizing the normal element y, we may assume $y = \operatorname{diag}(t_i)$, where the t_i lie in $C(X, \mathbf{R}^+)$ and $\prod t_i = 1$.

Let λ be a small positive number, to be determined later. By the partition property of Boolean spaces, there exists a finite disjoint clopen covering $\{K_\alpha\}$ for X, and points $x_\alpha \in K_\alpha$, so that for all i, $|t_i(x) - t_i(x)| < \lambda$ for all x in K_α. Let e_α be the characteristic function of K_α, so that R splits as a direct product. A rearrangement of a diagonal matrix will be unitarily equivalent (within $e_\alpha R$) to the original, so we may assume that for each α, the real numbers $\{t_i(x_\alpha)\}_{i=1}^n$ are already arranged in the order prescribed by Lemma 4. Now define the matrices

$$z = \operatorname{diag}(1, t_1, t_1t_2, \ldots, t_1t_2 \cdots t_{n-1}),$$

and w as defined in (a). Then $zw = y^{-1}wz$, so $y = wzw^{-1}z^{-1}$.

Now

$$\begin{aligned}\|z\| &= \max\{1, \|t_1\|, \|t_1t_2\|, \ldots\} \\ &\leqslant \max_\alpha\{1, |t_1(x_\alpha)|, |t_1t_2(x_\alpha)|, \ldots\} + (n-1)\lambda\|y\| \\ &\leqslant \max_\alpha\{t_1(x_\alpha), t_2(x_\alpha)\} + (n-1)\lambda\|y\| \\ &\leqslant \max\{\|t_1\|, \|t_2\|\} + (n-1)\lambda\|y\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\|z^{-1}\| &\leqslant \max\{\|t_1^{-1}\|, \|t_2^{-1}\|\} + (n-1)\lambda\|y^{-1}\|(\|y^{-1}\| + \lambda) \\ &\leqslant \|y^{-1}\| + 2(n-1)\lambda\|y^{-1}\|^2 \quad (\text{if } \lambda \leqslant \|y^{-1}\|).\end{aligned}$$

Choose $\lambda < \min\{\theta/2(n-1)\max\{\|y\|, \|y^{-1}\|^2\}, \|y^{-1}\|\}$.

Finally, if $f(x) = (\det y)^{-1/n}(x)$, then $(\det fy)(x) = 1$ for all x, and the last statement follows. □

Let $[a, b]$ denote the group commutator of invertible elements, $aba^{-1}b^{-1}$.

THEOREM 6. *Let X_n be a collection of extremely disconnected compact spaces, and set $R_n = M_nC(X_n)$. Define an AW^* algebra R to be the C^* sum of R_n. Then any invertible element b in R can be written in the form*

$$b = f[v, w][z, u] \tag{1}$$

where z is positive, u, v, and w are unitary and f is central. Further, $(ff^*)^{1/2}$ is independent of the choice of f, u, v, w, z in (1).

PROOF. We observe that b being invertible is the product of a unitary by a positive, so it suffices to show every unitary and every positive invertible element is central by a commutator.

Let e_n denote the identity of R_n. If b is unitary, be_n is unitary in R_n, so by the previous lemma there exist unitaries in R_n, v_n, w_n, so $be_n = f_n[v_n, w_n]$, where f_n is in the centre. Define $f = (f_n)$, $v = (v_n)$, and $w = (w_n)$; since the elements are all unitary, f, v, and w lie in the C^* sum, R, and clearly $b = f[v, w]$.

If b is positive, since b^{-1} lies in R, we have

$$\sup_n \{\|be_n\|, \|b^{-1}e_n\|\} = K < \infty.$$

Assume to begin with that for all n, for all x in X_n, $(\det be_n)(x) = 1$. By the preceding result, there exist z_n positive, and u_n unitary in R_n so that $be_n = [z_n, u_n]$, and $\|z_n\|, \|z_n^{-1}\| < 2K$. Thus $z = (z_n)$ belongs to R, and we are finished with this case.

Dropping the restriction on the determinants of be_n, we observe that for any x in X_n, for any positive y in R_n, $|(\det y)^{1/n}(x)| \leq \|y\|$. Thus $\|\det be_n\|^{1/n} \leq K$, and similarly, $\|\det b^{-1}e_n\|^{1/n} < K$. Thus the functions,

$$f = (\det b)^{1/n}\colon \dot{\bigcup} X_n \to \mathbf{R}^+, \quad \text{and} \quad f^{-1} = (\det b^{-1})^{1/n}\colon \dot{\bigcup} X_n \to \mathbf{R}^+$$

are bounded, so belong to $C(\beta(\dot{\bigcup} X_n))$, the centre of R. We see that for each n, $(\det fbe_n)(x) = 1$ for all x in X_n, so b can be written in the form $f^{-1}[z, u]$.

The final statement is an easy consequence of a determinant argument. □

If G is a group, then G^{ab} denotes G abelianized, that is, $G/[G, G]$,

THEOREM 7. *Let R be a finite AW^* algebra. Then $K_1(R) = \mathrm{GL}(1, R)^{\mathrm{ab}}$. If R is of type* I, *then $K_1(R) = C(X, \mathbf{C} - \{0\})$, where X is the maximal ideal space of the centre of R.*

PROOF. If R is of type II, then $R = M_2S$ for some ring S, and S must have 1 in the stable range [9, Theorem 3]. Now for rings satisfying 1-stable range, the natural maps, $\mathrm{GL}(m, S)^{\mathrm{ab}} \to \mathrm{GL}(m + 1, S)^{\mathrm{ab}}$ are surjective for all m, and injective for $m > 1$. Since $K_1(R) = K_1(S) = (\lim \mathrm{GL}(m, S))^{\mathrm{ab}}$, we see $K_1(R) = K_1(S) = \mathrm{GL}(2, S)^{\mathrm{ab}} = \mathrm{GL}(1, R)^{\mathrm{ab}}$.

If R is of type I, then R is a C^* sum as described in Theorem 6. It is an immediate consequence of that theorem (and the use of the determinant-like functions described there), that the map $\mathrm{GL}(1, R)^{\mathrm{ab}} \to \mathrm{GL}(2, R)^{\mathrm{ab}}$ is an embedding, so as in the previous paragraph, must be an isomorphism, and hence again by that paragraph, $K_1(R) = \mathrm{GL}(1, R)^{\mathrm{ab}}$, and routine use of Theorem 6 shows $\mathrm{GL}(1, R)^{\mathrm{ab}}$ is isomorphic to the group of invertible elements of the centre.

Since K_1 commutes with finite products, and every finite AW^* algebra is a product of one of type I and one of type II, this completes the proof. □

This suggests the problem of determining K_1 for type II AW^* algebras, in

particular for factors. For W^* factors, the Fuglede-Kadison determinant [4] induces an onto homomorphism d: $K_1(R) \to \mathbf{R}^+$, and this appears to be an isomorphism. For *some* factors, obtainable from the C^* sums considered above, it is an isomorphism, and these factors have very much stronger properties. Before introducing them, we need results on when unitaries lift through homomorphisms.

LEMMA 8. *Let R be a ring, and I a two-sided ideal of R. Suppose R has* 1 *in its stable range.*

(a) *Every invertible element of R/I is the image of an invertible element, and R/I also satisfies* 1*-stable range.*

(b) *If R is a C^* algebra, and $I = I^*$ (e.g., if I is closed), then every unitary in R/I is the image of one from R.*

(c) *If R is as in* (b), *and satisfies unitary* 1*-stable range, then so does R/I.*

PROOF. (a) If $v + I$ is invertible in R/I, there exist r in R and i in I such that $vr + i = 1$; thus $vR + iR = R$. By the stable range condition, there exists t in R with $v + it$ invertible, and $v + it$ maps to $v + I$.

If $(a + I)R/I + (b + I)R/I = R/I$, there exists r, s in R and i in I so that $ar + bs + i = 1$. Thus $bR + (ar + i)R = R$, so there exists t in R with $b + (ar + i)t$ invertible. Hence $b + art + I$ is invertible in R/I.

(b) We first note that if $x^*x \in 1 + I$, then $(x^*x)^{1/2} \in 1 + I$: $(1 - (x^*x)^{1/2})(1 + (x^*x)^{1/2}) \in I$, but $1 + (x^*x)^{1/2}$ is invertible. Now, by (a), every unitary $v + I$ of R/I is the image of an invertible element x from R. Then $x(x^*x)^{-1/2}$ is a unitary in R, mapping to $v + I$.

(c) As in the proof of (a). □

Let I be an infinite collection of integers that is unbounded. Define the ring S to be the C^* sum of $\{M_n\mathbf{C} | n \in I\}$. Let M be any maximal ideal (two-sided) that is not a direct summand, i.e. M will contain central projections of S corresponding to a nonprincipal ultrafilter on I. As is well known, $R = S/M$ is a finite W^* factor of type II (not representable on a separable Hilbert space). Let J' denote the closure of the ideal generated by the minimal central idempotents; then J' is contained in M. Let J denote any closed prime ideal containing J'.

Claim. In S/J, every unitary scalar is a unitary commutator.

Pick the unitary scalar λ. Because I is unbounded, if for each n in I, we pick λ_n an nth root of unity as close as possible to λ, we find the sequence $\{\lambda_n\}_{n \in I}$ has λ as a limit point. Since S/J is prime, the centre is $\mathbf{C}$, and it easily follows that $\lambda + J = (\lambda_n) + J$. Since λ_n is an nth root of unity in $M_n\mathbf{C}$, there exist unitaries u_n, v_n in $M_n\mathbf{C}$ so that $\lambda_n = [u_n, v_n]$. Setting $u = (u_n)$, $v = (v_n)$, we see u, v lie in S, and $\lambda + J = [u, v] + J$.

Now we can prove the following:

(i) Every unitary in R is the product of at most two unitary commutators,

(ii) Every positive invertible element of R is a positive scalar times a multiplicative commutator,

(iii) The Fuglede-Kadison determinant induces an isomorphism

$$d\colon K_1(R) \to \mathbf{R}^+.$$

PROOF. (i), (ii) By Lemma 8, every unitary (positive invertible) in R is the image of one from S. Then (i) follows from the comment immediately above and the proof of Theorem 6, and (ii) is also an immediate consequence of the proof of Theorem 6.

Since every invertible element of R lifts to an invertible element of S, and the Fuglede-Kadison determinant is a group homomorphism from $\mathrm{GL}(1, R)$ to $\mathbf{R}^+$ that leaves invariant the positive scalars and sends the unitary scalars to 1, we see that d is an isomorphism $\mathrm{GL}(1, R)^{\mathrm{ab}} \to \mathbf{R}^+$.

R has the interesting property as well that it satisfies the invariant subspace problem, i.e. for all r in R, there exists a nonzero projection p such that $rp = prp$ (one can even assume that $p \sim 1 - p$). This follows because in any finite dimensional C^* algebra, for all t there exists a projection q such that $tr(1 - 2q) = 0,1$ and $tq = qtq$. Then the sequence of the q's in S has nonzero image (because the image of the sequence is equivalent to 1 minus itself).

II. Unitary decomposition versus 1-stable range. We have seen that unitary decomposition implies unitary 1-stable range. The converse does not hold as we will show $C(X)$ satisfies unitary decomposition if and only if X is an F-space and $\dim X \leqslant 1$, whereas $C(X)$ satisfies 1-stable range (as well as unitary 1-stable range) if and only if the latter condition only holds.

A compact (Hausdorff) space X is an F-space [**6**, §14] if every bounded continuous (real or complex) function defined on a cozero set of X may be extended continuously to X. Our notion of dimension for topological spaces comes from [**8**, p. 8], and agrees with the two dimensions $d(X)$, $d'(X)$ found in [**9**, p. 104] (see [**9**, Theorem 5, last equation] and [**8**, III.1]). We denote the dimension of X by $\dim X$. $C(X)$ will always denote the ring of complex-valued continuous functions on X.

LEMMA 9. *For compact X, $\dim X \leqslant 1$ if and only if for all onto *-homomorphisms $C(X) \to C(Y)$, every unitary of $C(Y)$ is the image of a $C(X)$-unitary.*

PROOF. By [**8**, III.2], spaces with $\dim X \leqslant 1$ are characterized by the statement: If Y is a closed subspace of X and $f\colon Y \to S^1$ is continuous, then f can be extended to a continuous function $X \to S^1$. Regarding S^1 as the unit circle of $\mathbf{C}$, the result follows. □

PROPOSITION 10 [**9**, Theorem 7]. *For a topological space X, $\dim X \leqslant 1$ if and only if $C(X)$ has 1 in its stable range.*

THEOREM 11. *Let X be a compact Hausdorff space.*

(a) *The ring $C(X)$ has 1 in its stable range if and only if it satisfies unitary 1-stable range.*

(b) *$C(X)$ has unitary decomposition if and only if X is an F-space, $\dim X \leqslant 1$.*

PROOF. (a) If 1 is in the stable range, by [7] $\dim X \leqslant 1$. Set $R = C(X)$, and suppose $fR + gR = R$ for some f, g in R. We see that for all x in X, $(|f| + |g|)(x) \neq 0$. Define the compact subspace V of X by

$$V = \{x \in X \mid |f(x)| = |g(x)|\}.$$

Neither f nor g vanishes at any point of V, so we may define a unitary h in $C(V)$ by $h(v) = f(v)/g(v)$. The restriction map $C(X) \to C(V)$ is onto (Tietze extension theorem), so by Lemma 9, there is a unitary k in $C(X)$ so $k/V = h$. If x lies in V, $(f + gk)(x) = 2f(x) \neq 0$, while if $x \notin V$, $|f(x)| \neq |g(x)|$, so $f(x) \neq -gk(x)$. Hence $f + gk$ never vanishes, so is invertible.

(b) Unitary decomposition implies 1-stable range so $\dim X \leqslant 1$. Pick f in $C(X, \mathbf{R}) \subseteq C(X)$. We may write $f = u|f|$ for some unitary u in $C(X)$. Then $u = -1$ on $\{x \in X \mid f(x) < 0\}$ and $u = 1$ on $\{x \in X \mid f(x) > 0\}$. By [6, 14.25] X is an F-space.

Now suppose $\dim X \leqslant 1$ and X is an F-space. Pick f in $C(X)$, and define $U = \{x \in X \mid f(x) \neq 0\}$. Set Y to be the closure of U. Define $u \in C(U)$ by $u(x) = (f/|f|)(x)$. Then u is a unitary of $C(U)$. As Y is a compact subspace of an F-space (so is itself one), u extends to v in $C(Y)$; necessarily v is unitary. As in the proof of (a), there exists a w in $C(X)$ that is unitary and agrees with v on Y. Now $f - w|f|$ vanishes on U and both f, $|f|$, vanish off U, whence $f = w|f|$. $\square$

The space $X = \{1/n\}_{n \in \mathbf{Z}} \dot{\cup} \{0\}$ is compact, totally disconnected ($\dim X = 0$) but does not have unitary decomposition ($f(x) = x$). There exist compact connected F-spaces of dim 1, and of higher dimension ([6, 14.26]).

III. Other results. I had asked whether AF C^* algebras satisfy 1-stable range. The referee supplied the affirmative answer.

THEOREM 12 [REFEREE]. *Let R be a C^* algebra that contains a dense $*$-subalgebra satisfying unitary decomposition. Then R has unitary 1-stable range.*

PROOF. Let S denote the dense $*$-subalgebra, and choose a_i, x_i in R so that $a_1x_1 + a_2x_2 = 1$, $\|a_i\| \leqslant 1$, $\|x_i\| \leqslant K$, say. For $\delta > 0$, find a_i', x_i' in S differing from their counterparts by less than δ in norm with $\|a_i'\| \leqslant \|a_i\|$. Then $\|\sum a_i'x_i' - 1\| \leqslant 2\delta(K + 1)$, so for δ small, $z = \sum a_i'x_i'$ is invertible. From the identity $\|z^{-1}\| \leqslant 1/(1 - \|z\|)$ if $\|1 - z\| < 1$, setting $x_i'' = x_i'z^{-1}$, we see $a_1'x_1'' + a_2'x_2'' = 1$ and

$$\|x_i''\| \leqslant (\|x_i'\|)/(1 - 2\delta(K + 1)) \leqslant K/(1 - 2\delta(K + 1)).$$

By unitary decomposition, there exist u_i unitaries in S with $a_i'u_i = (a_i'a_i'^*)^{1/2}$. By the method of proof of Lemma 1, $(|a_1'| + |a_2'|)^{-1} \leqslant 1/\max\|x_i\|^2$. There thus exists a unitary u so that $(a_1' + a_2'u)^{-1} \leqslant K^2/(1 - 2\delta(K + 1))^2$. But

$$\|a_1 + a_2u - (a_1' + a_2'u)\| \leqslant 2\delta,$$

whence

$$\|1 - (a_1 + a_2u)(a_1' + a_2'u)^{-1}\| < 4\delta K^2/(1 - 2\delta(K + 1))^2;$$

so if δ is sufficiently small, $a_1 + a_2u$ is invertible. □

A few other properties of a C^* algebra with 1 in the stable range:

(i) All matrix rings satisfy 1-stable range ([**9**, Theorem 3]).

(ii) All matrix rings are finite ($xx^* = 1$ implies $x^*x = 1$).

(iii) If P,Q are equivalent projections of a matrix ring, they are unitarily equivalent.

Questions. 1, 2. How do the properties of unitary decomposition and unitary 1-stable range behave under the operation of taking matrix rings?

3,4. If R is a C^* algebra satisfying unitary decomposition (1-stable range) do the maximal abelian subalgebras (masa's) also satisfy it? (This seems unlikely, in view of Theorem 12.) If R is finite, and the masa's satisfy either property, does R?

Apropos question 1, it is easy to show that if M_2R satisfies unitary decomposition (or merely $f = v(f^*f)^{1/2}$ for some v in M_2R, for all f), then all finitely generated right ideals of R are principal; consider the right ideal of M_2R generated by $\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right]$; it is readily checked $aR + bR = (aa^* + bb^*)^{1/2}R$, as is the case with von Neumann algebras (cf. [**3**, Theorem 2.2, Corollary 3]).

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