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IDEALS OF FINITE CODIMENSION IN $C[0, 1]$ AND $L^1(R)$

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ABSTRACT. We characterize the ideals of finite codimension in $C[0, 1]$ and $L^1(R)$.

Let M be a subspace of codimension 1 in a commutative Banach algebra with identity. It was shown in [**6**] and [**8**] that if each element x in M belongs to a maximal ideal I_x, *which may depend on* x, then M is itself a maximal ideal. This interesting result as first proved depended on the Hadamard Factorization Theorem: later proofs used a one-sided Liouville Theorem [**10**], [**1**, p. 51], [**2**, Problem 11, p. 111], [**4**, Lemma 32, p. 1043].

This characterization of maximal ideals was extended in [**11**] to algebras without identity. The main results were: (1) [**11**, Theorem 2]: Let A be a commutative Banach algebra with one generator. If a closed subspace M of codimension 1 in A has the property that each element in M belongs to a regular maximal ideal, then M is a regular maximal ideal. (The proof given in [**11**] is in error, but it can be corrected.) An example is given of an algebra with two generators in which this characterization does not hold. (2) [**11**, Theorems 4 and 5]. Let G be a locally compact Abelian group with $\hat{G}$ sigma-compact. Then a subspace M of codimension 1 in $L^1(G)$ which has the property that each element in M belongs to some regular maximal ideal must itself be a regular maximal ideal. (If $\hat{G}$ is not sigma-compact, then each Fourier transform vanishes and so each subspace of codimension 1 has the property which is meant to characterize maximal ideals.) This result is also a corollary of the more basic result below.

THEOREM 1. *Let A be a commutative Banach algebra with involution $x \to x^*$, satisfying $\hat{x}^* = \hat{x}^-$. Suppose that there is an element x_0 in A with $\hat{x}_0$ never zero. If M is a subspace* (*not a priori closed*) *of codimension* 1 *in A with the property that each element in M belongs to some regular maximal ideal, then M is a regular maximal ideal.*

PROOF. Replacing x_0 by $x_0x_0^*$, we may suppose that $\hat{x}_0$ is real-valued (and never zero). By hypothesis $\mathrm{sp}(x_0) \cap M = (0)$ and so there is a linear functional F (not a priori continuous) satisfying $F(M) = \{0\}$ and $F(x_0) = 1$.

We claim that x in M implies that x^* is in M. For suppose not, then

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$F(x^*) = a \neq 0$ and $x^* - ax_0$ is in M (since $F(x^* - ax_0) = 0$). Now $x + x^* - ax_0$ belongs to M and so has hat vanishing at some point s in the regular maximal ideal space of A; $2 \operatorname{Re} \hat{x}(s) - a\hat{x}_0(s) = 0$ and so a is real. Also $-x + x^* - ax_0$ belongs to M and so for some t, $-2i \operatorname{Im} \hat{x}(t) - a\hat{x}_0(t) = 0$, and so a is pure imaginary. Hence a is zero.

We claim that x in M implies that xx_0 is in M. First suppose that $\hat{x}$ is real-valued. Then $xx_0 - F(xx_0)x_0$ is in M and so has hat vanishing at some regular maximal ideal from which we see that $F(xx_0)$ is real. Then $ix + xx_0 - F(xx_0)x_0$ is in M and has hat vanishing at some point s; evaluating at s we get $F(xx_0) = 0$. For x in M, x^* is also in M and so $(x + x^*)x_0$ and $i(x - x^*)x_0$ are in M by the argument above. Hence $(x + x^*)x_0 - i(i(x - x^*)x_0) = 2xx_0$ is in M.

For any x, $F(xx_0) = F(x)F(x_0^2)$. To see this, $x - F(x)x_0$ is in M and so $(x - F(x)x_0)x_0$ is in M and so $F(xx_0 - F(x)x_0^2) = 0$.

If x and y are in M, then xy is in M. The identity $4xy = (x + y)^2 - (x - y)^2$ shows that it suffices to prove that x in M implies that x^2 is in M. First suppose that $\hat{x}$ is real-valued. Then $x^2 - F(x^2)x_0$ is in M and has a hat vanishing at some regular maximal ideal from which we see that $F(x^2)$ is real-valued. Then $ix + x^2 - F(x^2)x_0$ is in M; evaluating the hat at a point where it vanishes we see that $F(x^2) = 0$. For an arbitrary x in M, $(x - x^*)^2$, $(i(x - x^*))^2$, and $(x - x^*)(i(x - x^*))$ are in M by what we have shown above. Hence

$$4x^2 = (x + x^*)^2 + (x - x^*)^2 + 2(x + x^*)(x - x^*)$$

is in M.

For any x and y, $F(xy) = F(x)F(y)F(x_0^2)$. To show this, note that $x - F(x)x_0$ and $y - F(y)x_0$ are in M, so the product is in M. That is:

$$F((x - F(x)x_0)(y - F(y)x_0)) = 0$$

or

$$F(xy) - F(x)F(yx_0) - F(y)F(xx_0) + F(x)F(y)F(x_0^2) = 0.$$

Using the fact that for any z, $F(zx_0) = F(z)F(x_0^2)$, $F(xy) - F(x)F(y)F(x_0^2) = 0$.

Finally, the functional $G(x) = F(x_0^2)F(x)$ is seen from the above equation to be a (nonzero) multiplicative linear functional–and therefore a continuous linear functional–vanishing on M. Hence, M is a regular maximal ideal. Q.E.D.

The results for algebras without identities are a special case of the natural generalization of the codimension 1 theorem to subspaces of codimension 2. To see this, let M be a subspace of codimension 1 in an algebra A without identity. Note that if M has the property that each x in M belongs to a regular maximal ideal, then in A_e, the algebra obtained by the standard adjunction of an identity [9], M is of codimension 2 and has the property that each x in M belongs to two maximal ideals in A_e.

Thus, from the failure of the codimension 1 result in some Banach algebras without identity, we see that the natural generalization of the codimension 1 result of **[6]** and **[8]** does not always hold for codimension 2.[1] We give below two examples of important algebras in which this generalization is true.

THEOREM 2. *Let S be a compact subset of the real line. If M is a closed subspace of codimension n in $C(S)$ with the property that each function in M vanishes at at least n distinct points of S, then M is an ideal.*

PROOF. By translating S we may suppose that 0 is not in S.

The subspace $\mathrm{sp}(1, x, \dots, x^n)$ is $n + 1$ dimensional so there is a nonzero element q_0 in $\mathrm{sp}(1, x, \dots, x^n) \cap M$. By hypothesis, q_0 has at least n distinct zeros in S, and so, to within a scalar, $q_0(x) = (x - s_1)(x - s_2) \cdots (x - s_n)$ where the s_i are distinct points of S.

Suppose that f is a real-valued function in M. Then $f + iq_0$ is in M and therefore must vanish at n distinct points of S; since q_0 vanishes only at $s_1, \dots, s_n$, f must also vanish at these points.

The subspace $\mathrm{sp}(x^k, x^{k+1}, \dots, x^{k+n})$ intersects M in a nonzero polynomial q_k. By hypothesis q_k must have n roots $r_1, \dots, r_n$ in S, and also has zero as a root of multiplicity k. To within a scalar, $q_k(x) = x^k(x - r_1) \cdots (x - r_n)$. Because the roots r_i are real, q_k is real-valued, and by the argument of the second paragraph $\{r_1, \dots, r_n\} = \{s_1, \dots, s_n\}$.

The subspace M must therefore contain

$$B = \{P(x)(x - s_1)(x - s_2) \cdots (x - s_n) \colon P \text{ a polynomial}\}.$$

The set B is a selfadjoint subalgebra which separates all points other than the s_i. By the general Stone-Weierstrass Theorem, the closure $\mathrm{cl}(B)$ of B is $\{f\colon f(s_i) = 0, 1 \leq i \leq n\}$; since $\mathrm{cl}\, B \subseteq M$ and $\mathrm{cod}(\mathrm{cl}\, B) = n$, $M = \mathrm{cl}\, B$. Q.E.D.

Theorem 2 does not hold for an arbitrary $C(S)$ algebra, S compact, for if S contains a point s_0 not a G_δ, then any element in any subspace M contained in $\{f\colon f(s_0) = 0\}$ must have infinitely many zeros. Does Theorem 2 hold if each point in S is a G_δ? Does Theorem 2 hold for S the unit disk in R^2?

There is an interesting approximation theory corollary to Theorem 2.

COROLLARY 3. *Let P be a continuous linear projection on $C[0, 1]$ with n dimensional range. Suppose that P has the following interpolation property*:

(*) *For each f there are distinct points $s_1, \dots, s_n$, depending on f, where Pf interpolates to f, i.e., $Pf(s_i) = f(s_i)$, $1 \leq i \leq n$.*

Then there are fixed points $t_1, \dots, t_n$ and functions $f_1, \dots, f_n$ with

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[1] **Even an ideal of codimension 2 is not, in general, the intersection of two maximal ideals. It is not hard to show that in a Banach algebra with identity, if I is a (closed) ideal of codimension 2, then one of the two following disjoint possibilities holds:**

(i) I is the intersection of two maximal ideals, or

(ii) I is the intersection of a maximal ideal and the kernel of a (continuous) point derivation at that maximal ideal. Compare this with [1, Theorem 1.6.1, p. 66].

$$Pf = \sum f(t_j)f_j \quad \textit{and} \quad f_i(t_j) = \delta_{ij}.$$

PROOF. Let M be the kernel of $I - P$, cod $M = \dim R(P) = n$. By Theorem 2 there are points $t_1, \ldots, t_n$ with

$$\{f: f(t_1) = \cdots = f(t_n) = 0\} = \bigcap N(e_{t_i}),$$

the intersection of the kernels $N(e_{t_i})$ of the evaluation functionals at t_i.

Write $P(f) = \sum x_i^*(f)h_i$, where the x_i^* are continuous linear functionals, and $\mathrm{sp}(h, \ldots, h_n) = R(P)$. By a basic lemma [3, Lemma 10, p. 421], since

$$N(x_i^*) \supset N(e_{t_j}), \qquad x_i^* = \sum_j a_{ij}e_{t_j}.$$

Thus

$$Pf = \sum_j f(t_j)\Big(\sum_i a_{ij}h_i\Big) = \sum f(t_j)f_j.$$

Since P is a projection onto its n dimensional range, the f_i are linearly independent, and $f_i = Pf_i = \sum_j f_i(t_j)f_j$ implies that $f_i(t_j) = f_{ij}$. Q.E.D.

This result is false for the space $C_R[0, 1]$ of *real-valued* continuous functions on $[0, 1]$; see [5] for related problems.

THEOREM 4. *Let M be a closed subspace of codimension n in $L^1(R)$. Suppose that each f in M belongs to at least n regular maximal ideals. Then M is an ideal.*

PROOF. Let h be a function in $L^1(R)$ with the properties:

(1) $\hat{h}$ is real valued and never zero,

(2) $\hat{h}(x)x^j = \hat{g}_j$ for some g_j in L^1, $1 \leq j \leq n$.

For example, consider $h_0(x) = \exp(-x^2/2)$. For this function, $\hat{h}_0(x) = h_0(x)$ [7, p. 415] and g_j can be found by applying the inverse Fourier transform to the rapidly decreasing function $h_0(x)x^j$ (e.g., [7, p. 409]). For h satisfying (1) and (2), let N be the subspace in L^1 whose Fourier transform is the subspace $\hat{h}(x)\mathrm{sp}(1, x, \ldots, x^n)$. Because $\dim N = n + 1$, N contains a nonzero element w of M; by hypothesis, to within a scalar multiple $\hat{w}(x) = \hat{h}(x)(x - s_1) \cdots (x - s_n)$, with $s_1, \ldots, s_n$ distinct real numbers. If h_1 were another function satisfying (1) and (2), there would be a nonzero element w_1 of M with $\hat{w}_1(x) = \hat{h}_1(x)(x - t_1) \cdots (x - t_n)$, $t_1, \ldots, t_n$ distinct real numbers. Since $w + iw_1$ belongs to M, $\{t_1, \ldots, t_n\} = \{s_1, \ldots, s_n\}$. Let $q_0(x) = (x - s_1) \cdots (x - s_n)$ and let w_0 be the element of M having Fourier transform $\hat{w}_0(x) = \hat{h}_0(x)q_0(x)$. At this point we have seen that if h is a function satisfying (1) and (2), then the element whose Fourier transform is $\hat{h}(x)q_0(x)$ belongs to M.

Since $h_0 * h_0$ satisfies (1) and (2), the element $h_0 * w_0$, whose Fourier transform is $h_0^2(x)q_0(x)$, belongs to M.

Suppose that f in L^1 has a real-valued Fourier transform $\hat{f}$ and set $b = \|\hat{f}\|_\infty + 1$. Then $h_0 * f + bh_0$ is in L^1 and has Fourier transform $h_0(x)(\hat{f}(x) + b)$ which is real valued and never zero, and (2) is satisfied as well. The function $h_0 * (h_0 * f + bh_0)$ also satisfies (1) and (2) and so, by the argument above, the element whose Fourier transform is $h_0^2(x)(f(x) + b)q_0(x)$, namely $(h_0 * f + bh_0) * w_0$, belongs to M. Consequently $f * (h_0 * w_0)$ belongs to M.

For any function g in L^1, let $\tilde{g}$ be the function satisfying $\tilde{g}\hat{} = \hat{g}^-$. Since $(g + \tilde{g}) * (h_0 * w_0)$ and $i(g - \tilde{g}) * (h_0 * w_0)$ belong to M, so does $g * (h_0 * w_0)$. Thus M contains the closed ideal J generated by $h_0 * w_0$. Since the hull of J is the finite point set $\{s_1, \ldots, s_n\}$, which is therefore a set of spectral synthesis, J is the kernel of the hull of J [9, p. 86]: $J = \{g: \hat{g}(s_i) = 0, 1 < i < n\}$. Since $\operatorname{cod} J = \operatorname{cod} M = n$ and $M \supseteq J$, $M = J$. Q.E.D.

For what locally compact Abelian groups G does $L^1(G)$ have the property of $L^1(R)$ described in Theorem 4? For G the unit circle, the character group $\hat{G}$ is a subgroup of R, and the proof of Theorem 4 can be easily adapted for this case. Does Theorem 4 hold for $L^1(R^2)$?

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