LINEAR MAPS OF $\mathcal{C}^*$-ALGEBRAS PRESERVING THE ABSOLUTE VALUE

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Abstract. In order that a linear map of $\mathcal{C}^*$-algebras $\phi: A \to B$ preserve absolute values, it is necessary and sufficient that it be 2-positive and preserve zero products of positive elements: if $x$ and $y$ are positive in $A$, with $xy = 0$, then $\phi(x)\phi(y) = 0$.

The generalized Schwarz inequalities of Kadison and Choi are extended to the nonunital case.

1. Introduction. In [4], linear maps $\phi$ of $\mathcal{C}^*$-algebras which preserve the absolute value were characterized as $*$-homomorphisms $\psi$ followed by a map $x \to bx = b^{1/2}xb^{1/2}$, where $b$ is a positive element centralizing the image of $\psi$. This sequel to [4] has as its principal purpose the exploration of other characterizations of these maps. The main result is that a linear map $\phi: \mathcal{A} \to \mathcal{B}$ of $\mathcal{C}^*$-algebras preserves the absolute value if and only if $\phi$ is 2-positive and preserves zero products of positive elements: if $x$ and $y$ are positive in $\mathcal{A}$, and $xy = 0$, then $\phi(x)\phi(y) = 0$ in $\mathcal{B}$. This is the main burden of Theorem 2. 2-positivity is relaxed to positivity in the presence of lots of projections in $\mathcal{A}$ (Theorem 1).

As in [4], one of our principal tools is the result of S. Sherman [8] to the effect that if $\mathcal{A}$ is a $\mathcal{C}^*$-algebra, the second conjugate (or double-dual) space $(\mathcal{A}^d)^d$ of the underlying Banach space $\mathcal{A}$ has a natural structure of $W^*$-algebra in which $\mathcal{A}$ is $\sigma$-weakly dense. $(\mathcal{A}^d)^d$ can be represented concretely as the $\sigma$-weak closure of $\pi(\mathcal{A})$, if $\pi$ is the universal representation of $\mathcal{A}$, the direct sum $\bigoplus_\sigma \pi_\sigma$ of the cyclic representations $\pi_\sigma$ arising from the states (normalized positive linear functionals) $\sigma$ of $\mathcal{A}$ by the Gel'fand-Neïmark-Segal construction. The Hilbert space underlying $\pi$ we call the universal representation space of $\mathcal{A}$. For a swift and complete account of this, see Kadison's article [6].

Roughly, our proofs proceed by showing that our hypotheses persist from $\phi: A \to B$ to the map $\phi^d: A^d \to B^d$ (second transpose map), and that $\phi^{dd}(I)$ which for simplicity we call $\phi(I)$, or $b$, lies in the centre of $\phi(\mathcal{A}^{dd})$. We then compose $\phi^{dd}$ with $x \to b^{-1/2}x = b^{-1/2}xb^{-1/2}$, a routine made precise and explicit in [4], use known results, including spectral theory, to establish the desired properties of the composed map $\psi$, and climb back down.
Other principal tools are the main results of Kaplansky's paper [7], both the density theorem and the strong continuity of the continuous functional calculus, and Choi's generalized Schwarz inequality [1].

By-products of the investigation include nonunital versions of the Choi and Kadison generalized Schwarz inequalities (Corollaries 1 and 2), of Choi's result that a 2-positive unital Jordan map of $\mathcal{C}^*$-algebras is a $\ast$-homomorphism (Corollary 6), and of Kadison's result that a unital linear map of $\mathcal{C}^*$-algebras preserving absolute values on self-adjoint elements is a Jordan map (Corollary 7).

2. Notations and definitions; statement of main results. For generalities on $\mathcal{C}^*$-algebras and $W^*$-algebras (von Neumann algebras), see the books of J. Dixmier [2], [3].

If $\mathcal{A}$ is a $\mathcal{C}^*$-algebra, $\mathcal{A}^+ = \{ x^*x : x \in \mathcal{A} \}$ is a closed, convex, proper cone, linearly spanning $\mathcal{A}$. Every element $a$ of $\mathcal{A}^+$ has a unique square root $a^{1/2}$ in $\mathcal{A}^+$. If $x \in \mathcal{A}$, $|x| = (x^*x)^{1/2}$ is the absolute value of $x$. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is positive if $\phi(\mathcal{A}^+) \subseteq \mathcal{B}^+$, and 2-positive if the map $\phi \otimes \text{id}_2$ is positive on the $\mathcal{C}^*$-algebra $\mathcal{A} \otimes M_2(\mathbb{C})$ to $\mathcal{B} \otimes M_2(\mathbb{C})$. Here $M_2(\mathbb{C})$ is the $\mathcal{C}^*$-algebra of $2 \times 2$ complex matrices [9]. The $\mathcal{C}^*$-algebra $\mathcal{A}$ is unital if $\mathcal{A}$ has a unit element $I_\mathcal{A}$ or $I$. $\phi$ is a Jordan map ($\mathcal{C}^*$-homomorphism) if $\phi(x^2) = \phi(x)^2$ for all (selfadjoint) $x$ in $\mathcal{A}$. If $\mathcal{A} \subseteq L(\mathcal{H})$, the $\mathcal{C}^*$-algebra of all bounded linear operators on the Hilbert space $\mathcal{H}$, and if $\eta \in \mathcal{H}$, then $\omega_\eta$ is the positive linear functional $x \to \langle x\eta, \eta \rangle$ on $\mathcal{A}$.

Definition. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ of $\mathcal{C}^*$-algebras will be called disjoint if $xy = 0$ in $\mathcal{A}$ implies $\phi(x)\phi(y) = 0$ in $\mathcal{B}$.

**Theorem 1.** A 2-positive, disjoint linear map of $\mathcal{C}^*$-algebras preserves absolute values.

If the domain algebra is $AW^*$ or approximately finite, “2-positive” can be replaced by “positive”.

**Theorem 2.** For a linear map $\phi : \mathcal{A} \to \mathcal{B}$ of $\mathcal{C}^*$-algebras, the following conditions are equivalent:

(i) $\phi$ preserves absolute values;
(ii) $\phi$ is positive, and $\phi(I)\phi(a_1a_2) = \phi(a_1)\phi(a_2)$ for all $a_1, a_2 \in \mathcal{A}$;
(iii) $\phi$ is 2-positive and disjoint;
(iii)' $\phi$ is 2-positive, and disjoint on positive elements.

In (ii) and hereafter, $\phi(I)$ has the interpretation $\phi(I) \in \mathcal{B}^{dd}$.

3. Details, proofs. The proofs of the theorems will follow a series of lemmas, some of independent interest.

**Lemma 1.** Let $\mathcal{A}$ be a unital $\mathcal{C}^*$-algebra in which the linear span of the projections is norm-dense. Let $\mathcal{B}$ be a unital $\mathcal{C}^*$-algebra, and let $\phi : \mathcal{A} \to \mathcal{B}$ be positive, unital and disjoint. Then $\phi$ is $\ast$-homomorphic.
Proof. Since $\phi$ is a positive unital map, it is selfadjoint. If $e$ is a projection in $\mathcal{G}$, and $a \in \mathcal{G}$, write

$$a = ae + a(1 - e)$$

and

$$\phi(a) = \phi(ae) + \phi(a(1 - e)),$$

and since $[a(1 - e)]e = 0$,

$$\phi(a)\phi(e) = \phi(ae)\phi(e),$$

while since $[ae](1 - e) = 0$,

$$\phi(ae)(1 - \phi(e)) = 0$$

or

$$\phi(ae)\phi(e) = \phi(ae).$$

Thus, $\phi(ae) = \phi(a)\phi(e)$, for all $a \in \mathcal{G}$ and projections $e \in \mathcal{G}$, but then since the linear span of the projection is dense in $\mathcal{G}$,

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all } a, b \in \mathcal{G}.$$

Thus, $\phi$ is $*$-homomorphic. □

Lemma 2. For a positive linear map $\phi: \mathcal{G} \to \mathcal{B}$ of C*-algebras, the following are equivalent:

(i) $\phi$ preserves $| \cdot |$ on selfadjoint elements;

(ii) $\phi$ is disjoint on selfadjoint elements.

Proof. (i) $\Rightarrow$ (ii). Suppose $a_1, a_2$ are selfadjoint in $\mathcal{G}$, and $a_1a_2 = 0$. Then since $a_1a_2^* = 0$,

$$a_1^*a_1a_2^*a_2 = 0, \quad |a_1| |a_2| = 0,$$

and

$$||a_1| - |a_2|| = |a_1| + |a_2|.$$ 

But then

$$|\phi(|a_1| - |a_2|)| = |\phi(|a_1|) - \phi(|a_2|)| = \phi(|a_1|) + \phi(|a_2|),$$

so $\phi(|a_1|)\phi(|a_2|) = 0$, that is, $|\phi(a_1)| |\phi(a_2)| = 0$, so $\phi(a_1)\phi(a_2) = 0$.

(ii) $\Rightarrow$ (i). If $a = a^* \in \mathcal{G}$, let $a = a^+ - a^-$ be its canonical decomposition with $a^+ > 0, a^- \geq 0, a^+a^- = 0$. Then $\phi(a) = \phi(a^+) - \phi(a^-)$ with $\phi(a^+) > 0, \phi(a^-) > 0, \phi(a^-)\phi(a^-) = 0$, so $\phi(a^+) = \phi(a^+), \phi(a^-) = \phi(a^-)$, and finally,

$$|\phi(a)| = \phi(a^+) + \phi(a^-) = \phi(a^+ + a^-) = \phi(|a|). \quad □$$

Lemma 3. If $\phi: \mathcal{G} \to \mathcal{B}$ is a 2-positive, linear map of C*-algebras, then $\phi^{\dd}$:

$\mathcal{G}^{\dd} \to \mathcal{B}^{\dd}$ is 2-positive, and satisfies $\phi^{\dd}(x^*\phi^{\dd}(x)) < ||\phi||\phi^{\dd}(x^*x)$ for all $x \in \mathcal{G}^{\dd}$.

Proof. $\phi$ is positive, so bounded; normalizing, we suppose it contractive. Then $\phi^{\dd}$ is a positive contraction. Put $b = \phi^{\dd}(I)$, and let $P$ be the support.
projection of $b$. Then $b^{-1}$ exists as a (usually unbounded) positive, selfadjoint operator affiliated with $P \mathcal{B}^\dd P$, and $\psi(x) = b^{-1/2} \phi^\dd(x) b^{-1/2}$ is well defined in $P \mathcal{B}^\dd P = \mathcal{B} \subset \mathcal{B}^\dd$ for $x \in \mathcal{B}^\dd$. Moreover, $\psi$ so defined is positive and unital on $\mathcal{B}^\dd$, and in fact is 2-positive since if $I_2$ and $I_2$, are, respectively, the identity mapping and the identity element of $M_2(\mathbb{C})$, we have

$$

\psi \otimes I_2 = (b^{-1/2} \phi^\dd(x)b^{-1/2}) \otimes I_2
$$

$$

= (b^{-1/2} \otimes I_2)(\phi^\dd(x) \otimes I_2)(b^{-1/2} \otimes I_2),
$$

while $\phi^\dd$ is 2-positive because $\phi^\dd \otimes I_2 = (\phi \otimes I_2)^\dd$ is $\sigma$-weakly continuous on $\mathcal{B}^\dd \otimes M_2(\mathbb{C}) = (\mathcal{B} \otimes M_2(\mathbb{C}))^\dd$, and $\phi \otimes I_2$ is positive. Now Choi's generalized Schwarz inequality [1, Corollary 2.8] applies, so that for $x \in \mathcal{B}^\dd$, we have $\psi(x)^*\psi(x) < \psi(x^*x)$, or writing $\tilde{\phi}$ for $\phi^\dd$, $b^{-1/2}\tilde{\phi}(x)^*b^{-1/2} \phi(x)b^{-1/2} < b^{-1/2}\tilde{\phi}(x^*x)b^{-1/2}$, whence $\tilde{\phi}(x)^*b^{-1/2}\tilde{\phi}(x) < \tilde{\phi}(x^*x)$. Now since $0 < b < I$ in $\mathcal{B}$,

$$

\tilde{\phi}(x)^*\tilde{\phi}(x) = \lim_{n \to \infty} \tilde{\phi}(x)^*b \left( b + \frac{1}{n} I \right)^{-1} \tilde{\phi}(x)
$$

$$

\leq \lim_{n \to \infty} \tilde{\phi}(x)^* \left( b + \frac{1}{n} I \right)^{-1} \tilde{\phi}(x) = \tilde{\phi}(x)^*b^{-1/2}\tilde{\phi}(x)
$$

$$

< \tilde{\phi}(x^*x),
$$

for $x \in \mathcal{B}^\dd$. Returning to the original, possibly noncontractive $\phi$, we have the inequality claimed in the lemma. □

The next two corollaries are nonunital generalizations of the results cited.

**Corollary 2.** In addition to removing the unital restriction, treats not only selfadjoint but normal elements, as does Størmer's Theorem 3.1 in [10].

**Corollary 1 (Choi's generalized Schwarz inequality).** If $\phi: \mathcal{A} \to \mathcal{B}$ is a 2-positive linear map of $\mathcal{C}^*$-algebras, $\phi(x)^*\phi(x) < ||\phi||\phi(x^*x)$ for all $x \in \mathcal{A}$.

**Corollary 2 (Kadison's generalized Schwarz inequality).** If $\phi: \mathcal{A} \to \mathcal{B}$ is a positive linear map of $\mathcal{C}^*$-algebras, $|\phi(x)|^2 < ||\phi||\phi(x^*x)$ for all normal $x \in \mathcal{A}$.

**Proof.** The restriction of $\phi$ to a commutative sub-$\mathcal{C}^*$-algebra of $\mathcal{A}$ is completely positive, by Stinespring [9, Theorem 4]. Since every normal $x \in \mathcal{A}$ is contained in such a subalgebra, Corollary 2 now follows from Corollary 1. □

**Corollary 3.** If $\phi: \mathcal{A} \to \mathcal{B}$ is a 2-positive linear map of $\mathcal{C}^*$-algebras, $\phi^\dd$ is strongly continuous.

**Proof.** If $\eta$ is a vector in the universal representation space of $\mathcal{B}$, we have, if, as we may assume, $\phi$ is contractive,
\[ \|\phi^{dd}(x)\eta\|^2 = \langle \phi^{dd}(x^*) \phi^{dd}(x) \eta, \eta \rangle = \langle \phi^{dd}(x^* x) \eta, \eta \rangle = \omega_\eta\left(\phi^{dd}(x^* x)\right) = \left(\phi^d(\omega_\eta)\right)(x^* x) = \|x\xi\|^2 \]

for some \( \xi \) in the universal representation space of \( \mathcal{A} \) independent of \( x \in \mathcal{A}^{dd}. \)

**Lemma 4.** If \( \phi \) is 2-positive, and disjoint on positive elements, \( \phi \) is disjoint.

**Proof.**

\[ a_1 a_2^* = 0 \Leftrightarrow a_1^* a_1 a_2 a_2^* = 0 \Leftrightarrow |a_1|^2 |a_2|^2 = 0 \Rightarrow \phi(|a_1|^2) \phi(|a_2|^2) = 0. \]

Since by Lemma 3,

\[ \|\phi\| \phi(|a_1|^2) > |\phi(a_1)|^2, \quad |\phi(a_1)| |\phi(a_2)|^2 = 0, \]

so \( \phi(a_1) \phi(a_2^*) = 0. \)

**Corollary 4.** A 2-positive, Jordan map is disjoint.

**Proof.** Apply Lemmas 2 and 4.

**Corollary 5** (Choi [1, Corollary 3.2]). A 2-positive unital Jordan map of \( \mathcal{C}^* \)-algebras is a \( * \)-homomorphism.

**Proof.** If \( \phi : \mathcal{A} \to \mathcal{B} \) is 2-positive unital and Jordan, so is \( \phi^{dd} : \mathcal{A}^{dd} \to \mathcal{B}^{dd} \), by Lemma 3 and Corollary 3; by the previous corollary, \( \phi^{dd} \) is disjoint; by Lemma 1, \( \phi^{dd} \) is \( * \)-homomorphic: Therefore, so is its restriction \( \phi \).

**Remark.** This proof is neither simpler than Choi's original proof, nor independent of the main results of his paper [1], but see Corollary 6.

**Proof of Theorem 1.** Let \( \phi : \mathcal{A} \to \mathcal{B} \) be a 2-positive, disjoint linear map of \( \mathcal{C}^* \)-algebras. Then by Corollary 3, \( \phi^{dd} \) is strongly continuous, while, by Lemma 2, \( \phi \) preserves \( | \cdot | \) on selfadjoint elements. By Kaplansky's density theorem and the strong continuity of \( | \cdot | \) on bounded sets of selfadjoint operators, we see that \( \phi^{dd} \) preserves \( | \cdot | \) on selfadjoint elements of \( \mathcal{B}^{dd} \), so that (again Lemma 2) \( \phi^{dd} \) is disjoint on the selfadjoint part of \( \mathcal{B}^{dd} \). Especially, if \( e_i \) (\( i = 1, 2 \)) are projections in \( \mathcal{B}^{dd} \), and \( e_1 e_2 = 0 \), then \( \phi(e_1) \phi(e_2) = 0 \). (Where no confusion can result, we write \( \phi \) instead of \( \phi^{dd} \).) Then if \( a \in \mathcal{A}^{dd} \), and \( a = \sum \lambda_j e_j \), with \( \lambda_j \in \mathcal{C} \), the \( e_j \) pairwise orthogonal projections, \( \phi(a) = \sum \lambda_j \phi(e_j) \), with the \( \phi(e_j) \) disjoint and positive, so

\[ |\phi(a)| = \sum |\lambda_j| |\phi(e_j)| = \phi\left( \sum |\lambda_j| e_j \right) = \phi(|a|). \]

Thus by spectral theory, \( \phi^{dd} \) preserves \( | \cdot | \) on normal elements, so on (unital) commutative \( * \)-subalgebras. From the first part of the proof of Theorem 1 of [4], we can conclude that \( b = \phi^{dd}(I) \) commutes with each normal element of \( \phi^{dd}(\mathcal{A}^{dd}) \), so with all of \( \phi^{dd}(\mathcal{A}^{dd}) \), and that \( \psi \) defined on \( \mathcal{B}^{dd} \) by \( \psi(a) = b^{-1} \). \( \phi^{dd}(a) \) is \( * \)-homomorphic on commutative \( * \)-subalgebras of \( \mathcal{B}^{dd} \), so is a Jordan homomorphism.
Since $\psi$ is by Lemma 3 a 2-positive map as well as Jordan-homomorphic, and unital to $\text{Supp } b \otimes \text{Supp } b$, Corollary 5 shows it $\ast$-homomorphic. Then $\phi = b \psi = b^{1/2} \psi b^{1/2}$ preserves absolute values. This proves the first statement of Theorem 1.

If $C$ is unital, and the set of linear combinations of orthogonal families of projections in $\mathcal{E}$ is norm-dense in $\mathcal{E}$, and if $\phi: \mathcal{E} \to \mathcal{B}$ is positive and disjoint, then we need not lift the argument above to $\mathcal{B}^\dd$, but argue directly in $\mathcal{E}$ as above that $b = \phi(I)$ centralizes $\phi(\mathcal{E})$ in $\mathcal{B}$, so

\[
\psi = b^{-1} \phi: \mathcal{E} \to \text{Supp } b \otimes \text{Supp } b
\]

(see [4, Theorem 2]), is positive, unital and disjoint. Now Lemma 1 shows that $\psi$ is $\ast$-homomorphic, so $\phi = b \psi$ preserves absolute values. This proves the second statement of Theorem 1. \[\square\]

**Lemma 5.** If $\mathcal{E}$ is a unital ring, $\mathcal{B}$ a ring, and $\phi: \mathcal{E} \to \mathcal{B}$ is an additive map satisfying $\phi(I)\phi(x^2) = \phi(x)^2$ for all $x$ in $\mathcal{E}$, then $\phi(I)$ centralizes $\phi(\mathcal{E})$.

**Proof.**

\[
\phi(I)\phi((I + x)^2) = \phi(I)\phi(I + 2x + x^2) = \phi(I)(\phi(I) + 2\phi(x)) + \phi(x)^2
\]

\[
= \phi(I)^2 + 2\phi(I)\phi(x) + \phi(x)^2;
\]

but this is

\[
(\phi(I) + \phi(x))^2 = \phi(I)^2 + \phi(I)\phi(x) + \phi(x)\phi(I) + \phi(x)^2,
\]

so $\phi(I)\phi(x) = \phi(x)\phi(I)$ for all $x \in \mathcal{E}$. \[\square\]

**Proof of Theorem 2.** That (i) $\Rightarrow$ (ii) was established in Theorem 2 of [4].

(ii) $\Rightarrow$ (i). Because $\phi^\dd$ is $\sigma$-weakly continuous, and multiplication is separately continuous in the $\sigma$-weak topology on $\mathcal{B}^\dd$ and $\mathcal{B}^\dd$, the identity (ii) persists for $\phi^\dd$, with $a_1, a_2 \in \mathcal{B}^\dd$. Then Lemma 5 applies, and $b = \phi(I)$ centralizes $\phi^\dd(\mathcal{B}^\dd)$. It then follows from (ii) that $\psi = b^{-1} \phi^\dd$ is $\ast$-homomorphic on $\mathcal{B}^\dd$ so $\phi^\dd$ preserves absolute values, as does its restriction $\phi$.

(i) $\Rightarrow$ (iii). That (i) $\Rightarrow$ $\phi$ completely positive and disjoint follows from Theorem 1 of [4].

(iii) $\Rightarrow$ (i). This is Theorem 1.

(iii) $\Rightarrow$ (iii)'. Trivial.

(iii)' $\Rightarrow$ (iii). This is Lemma 4. \[\square\]

**Corollary 6 (The nonunital version of Corollary 5).** A 2-positive Jordan map of $C^*$-algebras is a $\ast$-homomorphism.

**Proof.** Let $\phi: \mathcal{E} \to \mathcal{B}$ be such a map. Then by Corollary 4, $\phi$ is 2-positive and disjoint, so, by Theorem 2, $\phi$ preserves absolute values. Then, by Theorem 2 of [4], $\phi^\dd = \phi(I)\psi$, where $\psi$ is a unital $\ast$-homomorphism of $\mathcal{B}^\dd \to \text{Supp } \phi(I) \otimes \text{Supp } \phi(I)$. But $\phi^\dd$ is a Jordan map, so $\phi(I) = \phi^\dd(I)$ is
a projection: namely, \( \text{Supp} \phi(I) \). Thus \( \phi^{dd} = \psi \), and its restriction \( \phi \) is \(*\)-homomorphic. \( \square \)

**Corollary 7 (Nonunital version of [5, Theorem 6]).** If \( \phi : \mathcal{A} \rightarrow \mathcal{B} \) is a linear map of \( \mathcal{C}^*\)-algebras which preserves absolute values on selfadjoint elements, then there exist a unique Jordan map \( \psi : \mathcal{A} \rightarrow \mathcal{B}^{dd} \) and a unique positive element \( b \) of \( \mathcal{B}^{dd} \) supported on \( \bigcup_{a \in \mathcal{A}} \text{range} \phi(a) \) and centralizing \( \phi(\mathcal{A}) \) such that \( \phi(x) = b \psi(x) \) for all \( x \in \mathcal{A} \).

**Proof.** If \( \phi : \mathcal{A} \rightarrow \mathcal{B} \) preserves \( \cdot \) on selfadjoint elements, so does \( \phi^{dd} : \mathcal{A}^{dd} \rightarrow \mathcal{B}^{dd} \), as in the proof of Theorem 1, so by Lemma 2, \( \phi^{dd} \) preserves disjointness on selfadjoint elements. Let \( \mathcal{D} \) be a maximal commutative \(*\)-subalgebra of \( \mathcal{A}^{dd} \). Then \( I \in \mathcal{D} \). By [9, Theorem 4], the restriction of \( \phi^{dd} \) to \( \mathcal{D} \) is completely positive, so 2-positive; therefore it preserves absolute values, by Lemma 4 and Theorem 1. Now the first part of the proof of Theorem 1 in [4] shows that \( \phi(I) \) centralizes \( \phi^{dd}(\mathcal{D}) \), so \( \phi(I) \) commutes with \( \phi^{dd}(u) \) for all unitary \( u \) in \( \mathcal{B}^{dd} \), hence with all \( \phi^{dd}(x) \), \( x \in \mathcal{A}^{dd} \). Now with \( b = \phi(I) \) and \( P = \text{Supp} b \),

\[ \psi = b^{-1/2} \phi^{dd} = b^{-1/2} \phi^{dd}(-) b^{1/2}, \]

mapping \( \mathcal{A}^{dd} \) unitally into \( P \mathcal{B}^{dd} P \) satisfies the hypotheses of Theorem 5 of [5], so is a Jordan map, and \( \phi = b \psi \). This proves the existence claim.

The uniqueness claim can be proved following the uniqueness proof of [4], Theorem 2, or can be inferred from [4], Theorem 2 by restriction to commutative subsystems. \( \square \)

**Remark.** The transposition map on \( \mathcal{M}_2(\mathbb{C}) \) is unital, positive, and disjoint on positive elements, but is not disjoint. This shows that “2-positive” cannot be weakened to “positive” in (iii)’.

**Problem.** Can “2-positive” be replaced by “positive” in the first part of Theorem 1?

Finally, we note a further, easily proved relation between the properties “Jordan” and “\( | \cdot | \)-preserving” for linear maps: A Jordan homomorphism of \( \mathcal{C}^*\)-algebras which preserves absolute values is a \(*\)-homomorphism. In fact, if \( \psi : \mathcal{A} \rightarrow \mathcal{B} \) is such a map,

\[ \psi(a^*a) = \psi(|a|^2) = \psi(|a|)^2 = | \psi(a) |^2 = \psi(a^*) \psi(a). \]

A polarization completes the proof.

**References**


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