THE SOLVABILITY OF OPERATOR EQUATIONS WITH ASYMPTOTIC QUASIBOUNDED NONLINEARITIES

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Abstract. We study the solvability of operator equations involving quasi-bounded and asymptotically quasibounded nonlinear perturbations of linear Fredholm operators.

1. Let $X$ and $Y$ be Banach spaces, $L: X \to Y$ a linear Fredholm map of nonnegative index and $N: X \to Y$ a compact map. The operator equation of the form

$$Tx = Ax + Nx = f$$

has been extensively studied by many authors in recent years. Under various growth conditions on $N$, the surjectivity of $T$ has been proven in a number of papers (see [4], [5], [7] and the references therein).

Alternatively, beginning with a paper of Landesman and Lazer [6], much work has been done on the solvability of equation (1) for a certain range of values of $Pf$, where $P$ is the projection of $Y$ on the cokernel of $A$. Using the stable homotopy arguments, Nirenberg [9], [10], Berger [1], Mawhin [8], Podolak [11], Borisovich, Zvyagin and Sapronov [2] and others have studied equation (1). The alternative method has also been used to study equation (1) (with noncompact $N$ too) in a series of papers by Cesari and his coworkers, Fučík, Kučera and Nečas [5], and many others (cf. the survey paper by Cesari [3] and the monograph by Berger [1] for contributions of other authors). In all these papers (except in [2], [7], [11]) $N$ is assumed to have less than linear or linear growth.

In [2] and [11] the authors have studied equation (1) under the assumption that $N$ is asymptotically linear or asymptotically Lipschitz (i.e., $B$ in Definition 1 below is a Lipschitz map), respectively. In a series of papers Mawhin (cf. [7], [8]) has studied equation (1) with $f \in R(A)$ involving certain quasibounded maps $N$ using his coincidence degree.

In this paper we study the surjectivity of $T$ with $N$ either quasibounded or asymptotically quasibounded as defined below. Moreover, in case when the index of $A$, $i(A)$, is zero we provide a new growth condition on $PN|_{\ker A}$ that insures the solvability of equation (1) with these types of nonlinearities $N$. In the proofs of our main results we use a special case of the degree theory for

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compact perturbations of nonlinear $C^1$-Fredholm maps as developed in [2] or, equivalently, the stable homotopy arguments since for our map $T$ this degree can be defined in terms of elements of the stable homotopy group $\pi_{n+m}(S^m)$ (see [1], [2], [9]).

2. Set $X_1 = \ker A$ and $Y_2 = A(X)$. Since $A$ is Fredholm, $\dim X_1 = n < \infty$ and $Y_2$ is closed we have the following direct sum decompositions: $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ with $\dim Y_1 = m < \infty$ and $\text{ind}(A) = n - m > 0$. Define a new norm on $X$ by

$$\|x\|_1 = \max(\|x_1\|, \|x_2\|),$$

where $x = x_1 + x_2$ with $x_i \in X_i$, $i = 1, 2$. Let $P: Y \to Y_1$ be a linear continuous projection onto $Y_1$, $H$ be the inverse of the linear homeomorphism $A|_{X_2}$: $X_2 \to Y_2$ and $\alpha = \|H\|.$

**Theorem 1.** Suppose that for a given $f$ in $Y$ the following conditions hold:

(1) There exist constants $M_f > 0$ and $N_f > 0$ such that $PN(x_1 + x_2) - hf_1 \neq 0$ for $\|x_2\| \leq r$, $r > N_f$, $\|x_1\| > rM_f$ and $t \in [0, 1]$;

(2) $M = H(I - P)N$ is quasibounded, i.e.,

$$|M| = \limsup_{\|x\|_1 \to \infty} \frac{\|Mx\|}{\|x\|_1} < \infty$$

and $|M|\max\{1, M_f\} < 1$;

(3) the stable homotopy class $\eta_p$ of $PN|_{S^p_{n-1}}: S^p_{n-1} \to Y_1|\{0\}$, $\rho > rM_f$, is nontrivial, where $S^p_{n-1} \subset X_1$ is a sphere of radius $\rho$.

Then equation (1) is solvable for this $f$.

**Proof.** Let $\varepsilon > 0$ be small. By (2) there exists $R > N_f$ such that

$$\|Mx\| = \|H(I - P)Nx\| < (|M| + \varepsilon)\|x\|_1$$

for all $\|x\|_1 > R$. Moreover, there exists an $r > R$ such that $Ax + t(I - P)Nx - tf_1 \neq 0$ for all $x = x_1 + x_2$ with $\|x_1\| < rM_f$ and $\|x_2\| = r$ and $t \in [0, 1]$. If not, then for each $r > R$ there exist $t \in [0, 1]$ and $x$ with $\|x_1\| < rM_f$ and $\|x_2\| = r$ such that $Ax_2 + t(I - P)Nx - tf_2 = 0$, and therefore

$$\|x_2\| < \|H(I - P)Nx\| + \alpha\|f_2\| \leq (|M| + \varepsilon)\|x\|_1 + \alpha\|f_2\|,$$

or

$$1 < \frac{1}{r}(\|M| + \varepsilon)\|x\|_1 + \frac{\alpha}{r}\|f_2\| \leq (|M| + \varepsilon)\max\{1, M_f\} + \frac{\alpha}{r}\|f_2\|.$$ 

Passing to the limit as $r \to \infty$, we obtain $1 < (|M| + \varepsilon)\max\{1, M_f\}$ which is in contradiction with condition (2) for $\varepsilon$ small enough. Hence, an $r$ with the above property exists.

Next, we define $D = \{x = x_1 + x_2 \in X | \|x_1\| < rM_f, \|x_2\| < r\}$ with $r$ chosen as above, and define the homotopy $H: [0, 1] \times D \to Y$ by

$$H(t, x) = (Ax + t(I - P)Nx - tf_2, PN(x_1 + tx_2) - tf_1).$$
We claim that $H(t, x) \neq 0$ for $t \in [0, 1]$ and $x \in \partial D$. Indeed, if $x \in \partial D$ is such that $\|x_2\| < r$, then $\|x_1\| = rM_0$ and by (1), $PN(x_1 + tx_2) - t\phi \neq 0$ for all $t \in [0, 1]$. If $x \in \partial D$ is such that $\|x_1\| < rM_0$, then $\|x_2\| = r$ and $Ax + t(I - P)Nx - t\phi \neq 0$ for all $t \in [0, 1]$. Thus, by the homotopy theorem in [2],
\[
\deg(A + N - f, \overline{D}, 0) = \deg(H_0, \overline{D}, 0) = \eta_r,
\]
which, by the solvability property of this degree, implies that $Ax + Nx = f$ for some $x \in D$. □

To treat a larger class of nonlinear maps $N$, we need:

**Definition 1.** A map $N: X \to Y$ is said to be *asymptotically quasibounded* if there exists a nonzero continuous quasibounded map $B: X \to Y$, i.e.,
\[
|B| = \limsup_{\|x\| \to \infty} \frac{\|Bx\|}{\|x\|} < \infty
\]
such that
\[
(A) \quad \lim_{R \to \infty} \frac{N(Rx)}{R} = B(x) \text{ uniformly on bounded sets in } X.
\]
Such maps with $B$ Lipschitz have been studied by Podolak [11].

**Theorem 1** admits the following extension:

**Theorem 2.** Suppose that $N$ satisfies condition (A) and that $B$ is continuous, satisfies conditions (1) and (3) of Theorem 1 for $f = 0$ and that the following condition holds:

\[
(2') K = H(I - P)B \text{ is quasibounded, i.e.,}
\]
\[
|K| = \limsup_{\|x\| \to \infty} \frac{\|Kx\|}{\|x\|} < \infty
\]
and $|K| \max\{1, M_0\} < 1$.

Then equation (1) is solvable for each $f$ in $Y$.

**Proof.** Since for each $f$ in $Y$, $Nf = Nx - f$ satisfies condition (A) with the same $B$, it is sufficient to consider the case $f = 0$. Define
\[
\overline{D} = \{ x = x_1 + x_2 \in X \mid \|x_1\| < rM_0, \|x_2\| < r \},
\]
where $r$ is chosen as in Theorem 1 using property (2') of $K$. For $R > 0$, define the map $H_R: \overline{D} \to Y$ by
\[
H_R(x) = (1/R)(A(Rx) + (I - P)N(Rx), PN(Rx))
\]
and the homotopy $H: [0, 1] \times \overline{D} \to Y$ by
\[
H(t, x) = (Ax + t(I - P)Bx, PB(xx + tx_2)).
\]
By our choice of $r$ we know that $H(t, x) \neq 0$ for $t \in [0, 1]$ and $x \in \partial D$. Clearly, if $x \in X$ is a solution of equation (1), then $u = x/R \in D$ is a solution of $H_R(u) = 0$ for $R$ sufficiently large, and conversely. Moreover, $H_R(x) \neq 0$ uniformly for $x \in D$ since $H(1, \cdot)$ is a proper map. In view of this, it follows that for sufficiently large $R$, $H_R(x) \neq 0$ on $\partial D$ and
\[
F_R(t, x) = H(1, x) + t(H_R(x) - H(1, x)) \neq 0
\]
for \( t \in [0, 1] \) and \( x \in \partial D \). The compactness of \( N \) and condition (A) imply that \( B \) is compact and consequently

\[
F_R(t, x) = Ax + (1 - t)Bx + tN(Rx)/R
\]

is an admissible homotopy on \([0, 1] \times \partial D\) (cf. (4.2) in [2]). Hence,

\[
\deg(H_R, \partial D, 0) = \deg(H(1, \cdot), \partial D, 0) = \deg(H(0, \cdot), \partial D, 0) = \eta
\]

which implies that the equation \( H_R(x) = 0 \) is solvable in \( D \). □

**Remark.** When \( A \) is asymptotically linear, i.e., \( A(x) = B(x) + w(x), x \in X \), for some continuous and linear map \( B: X \rightarrow Y \) with \( w(x)/\|x\| \rightarrow 0 \) as \( \|x\| \rightarrow \infty \), then \( N \) is quasibounded with \( \|N\| = \|B\| \). Hence, Theorem 1 extends Theorem 4.5 in [2], which is, on the other hand, an abstract extension of some results of Nirenberg [9] involving everywhere bounded nonlinearities \( N \). Other extensions of Nirenberg’s results to sublinear or quasibounded nonlinearities are given in [1], [4], [5], [7], [8] (cf. [1] for other references).

**Remark.** If \( B \) in condition (A) is Lipschitz, i.e., \( \|Bx - By\| \leq k\|x - y\| \) for all \( x, y \in X \) and some small \( k > 0 \), then condition (1) in Theorem 2 can be replaced by the following easier to verify condition of Podolak [11]:

\[
(1') \|P_N(a \cdot x_0)\| \geq b \text{ for some positive } b \text{ and all } a \in R^n \text{ with } \|a\| = 1,
\]

where \( x_0 = \{x_{01}, \ldots, x_{0n}\} \) is a fixed basis for \( \ker A \) of unit vectors and

\[
a \cdot x_0 = a_1x_{01} + \cdots + a_nx_{0n}.
\]

In this sense Theorem 2 extends Theorem 1 in [11].

Let us now look at a new condition on \( P_N|_{X_1} \) which implies that \( \deg(P_N|_{X_1}, B(0, r), 0) \neq 0 \) with \( B(0, r) \subset X_1 \). Suppose that \( X \) and \( Y \) are such that there exist a map \( J: X_1 \rightarrow Y_1^* \) and a continuous and odd map \( G: X_1 \rightarrow Y_1 \) with \( Gx \neq 0 \) for \( x \neq 0 \) and \( (Gx, Jx) = \|Gx\| \cdot \|Jx\| \) for all \( x \in X_1 \). This is always so if \( Y = X \) or \( Y = X^* \). Indeed, if \( Y_1 = X_1 \), as \( G \) and \( J \) we can take the identity and the normalized duality map, respectively; while, if \( Y_1 = X_1^* \) as \( G \) and \( J \) we can take the normalized duality map and the identity, respectively. The condition in question is:

\[
(4) \|P_Nx\| + (P_Nx, Jx)/\|Jx\| > 0 \text{ for } x \in \partial B(0, \rho), \rho \geq rM_f.
\]

**Corollary 1.** Let \( A \) and \( N \) satisfy conditions (1) and (2) of Theorem 1. Then, if condition (4) holds for all \( \rho \geq rM_f \) and the index of \( A \) is zero, equation (1) is solvable.

**Proof.** By Theorem 1 it suffices to show that \( \deg(P_N, B(0, \rho), 0) \neq 0 \), where \( P_N \) is restricted to \( \bar{B}(0, \rho) \). Define the homotopy \( H: [0, 1] \times \bar{B}(0, \rho) \rightarrow Y_1^* \) by \( H(t, x) = tP_Nx + (1 - t)Gx \). Then \( H(t, x) \neq 0 \) for \( t \in [0, 1] \) and \( x \in \partial B \). If not, then \( tP_Nx + (1 - t)Gx = 0 \) for some \( t \in [0, 1] \) and \( x \in \partial B \). Since \( t \neq 0,1 \), we have

\[
\|P_Nx\| + \frac{(P_Nx, Jx)}{\|Jx\|} = \frac{1 - t}{t} \|Gx\| - \frac{1 - t}{t} \frac{(Gx, Jx)}{\|Jx\|} = 0
\]
in contradiction with condition (4). By the oddness of $G$ we obtain:
\[
\deg(PN, B(0, \rho), 0) = \deg(G, B(0, \rho), 0) \neq 0.
\]

Similarly, using Theorem 2, we obtain:

**Corollary 2.** Let $K$ be asymptotically quasibounded and $B$ satisfy conditions (1) and (2') of Theorem 2 with $f = 0$. Then, if ind $A = 0$ and $PB$ satisfies condition (4) for $f = 0$, equation (1) is solvable for each $f$ in $Y$.

Under a somewhat stronger condition than (4), we have:

**Theorem 3.** Let $X$ and $Y$ be Banach spaces with dim $X = \dim Y < \infty$ and let $T: X \to Y$ be continuous and satisfy
\[
(5) \quad \|Tx\| + \frac{(Tx, Jx)}{\|Jx\|} \to \infty \text{ as } \|x\| \to \infty, \text{ where } J \text{ and } G \text{ are as above.}
\]
Then $T(X) = Y$.

**Proof.** Let $f$ in $Y$ be fixed. By condition (5) there exists an $r_f > 0$ such that
\[
\|Tx - tf\| > 0 \quad \text{for } \|x\| = r_f, \quad t \in [0, 1]
\]
and
\[
\|Tx\| + \frac{(Tx, Jx)}{\|Jx\|} > 0 \quad \text{for } \|x\| = r_f.
\]
The first inequality implies that
\[
\deg(T - f, B(0, r_f), 0) = \deg(T, B(0, r_f), 0),
\]
which is nonzero by the second inequality as shown in Corollary 1. Hence, $Tx = f$ is solvable.

**Remark.** Along similar lines one can show that if $T: X \to X$ is continuous and compact (or condensing) and $I - T$ satisfies condition (5), then $(I - T)(X) = X$ (the proof will appear in a forthcoming paper by the author).

Condition (5) for $PN$ clearly holds if $PN$ is coercive on $X_1$, i.e.,
- if $(PNx, Jx)/\|Jx\| \to \infty$ as $\|x\| \to \infty$, $x \in X_1$, or
- if $(PNx, Jx) > c_1\|Jx\|$ for all $x \in X_1$ and some $c_1 > 0$ and $\|PNx\| \to \infty$ as $\|x\| \to \infty$, $x \in X_1$, and, in particular,
- if $\|PNx\| > c_2\|x\|^k$ for all $x \in X_1$ and some $c_2 > 0, k > 0$.

The last condition holds if $N$ is $k$-homogeneous. Indeed, since $\|PNx\| \neq 0$ for $x \in \partial B(0, r) \subset X_1$,
\[
a = \min\{\|PNx\| \mid \|x\| = r\} > 0
\]
and $\|PNx\| > (a/r^k)\|x\|^k$ for all $\|x\| > r$.

In view of the above discussion, we have the following special case of Theorem 2.1 in [8]:

**Theorem 4.** Let $A: D(A) \subset X \to Y$ be a linear Fredholm map of index zero and $N: D \subset X \to X$ a continuous compact map, where $D$ is open and bounded. Suppose that
- (i) $Ax \neq \lambda Nx$ for $x \in D(A) \cap \partial D$ and $\lambda \in (0, 1)$;
(ii) \( PNx \neq 0 \) for each \( x \in \ker A \cap \partial D \);
(iii) for some isomorphism \( L : Y_1 \to X_1 \),

\[
\|LPNx\| + \frac{(LPNx, Jx)}{\|Jx\|} > 0 \quad \text{for} \quad x \in \partial D \cap X_1
\]

with \( J \) the normalized duality map from \( X_1 \) to \( 2^{X_1} \).

Then the equation \( Ax - \lambda Nx = 0 \) has at least one solution in \( D \) for each \( \lambda \in [0, 1] \).

**Proof.** It suffices to show (cf. [8]) that \( \deg (LP|_{X_1}, D \cap X_1, 0) \neq 0 \). But, this follows from condition (iii) as in Corollary 1 since \( I \) is odd. □

**Remark.** The above results could be proven by using the homotopy

\[
H(t, x) = (x^2 + tH(I - P)Nx - tf_2, PN(x_1 + tx_2) - tf_1)
\]

instead. Hence, it is sufficient to require that the map \( H(I - P)N : X \to X \) be compact or condensing. The same observation holds for Theorem 2 with \( N \) replaced by \( B \). Moreover, Theorem 2 of Podolak [11] can be shown to be valid for the nonlinearities considered in our Theorem 2.

**References**


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