THE COHOMOLOGY OF THE PROJECTIVE n-PLANE\(^1\)

WILLIAM A. THEDFORD

Abstract. An H-space is a topological space with a continuous multiplication and an identity element. In this paper \(X\) has the homotopy type of a countable CW-complex with integral cohomology of finite type and primitively generated \(k\)-cohomology, \(k\) a field. The projective \(n\)-plane of \(X\) is denoted \(XP(n)\). The main results of this paper are: Theorem 1 which states that \(H^*(XP(n)) = N \oplus S\) where \(N\) is a truncated polynomial algebra over \(k\) and \(S\) is a trivial \(k\)-ideal, and Theorem 2 which considers the case \(k = Z(p)\) and states that \(H^*(XP(n)) = \tilde{N} \oplus S\) where \(\tilde{N}\) is a truncated polynomial algebra on generators in even dimensions and \(S\) is an \(A(p)\)-subalgebra of \(H^*(XP(n))\) so that an \(A(p)\)-algebra structure can be induced on \(N\). These theorems extend results by A. Borel, W. Browder, M. Rothenberg, N. E. Steenrod, and E. Thomas.

0. Introduction. An H-space is a topological space with a continuous multiplication and an identity element. In [10] Stasheff defined the projective \(n\)-plane of an H-space. In this paper \(X\) has the homotopy type of a countable CW-complex with integral cohomology of finite type and primitively generated \(k\)-cohomology, \(k\) a field. The two main results of this paper pertain to the cohomology of the projective \(n\)-plane, \(XP(n)\), of an H-space, \(X\). Theorem 1 states that

\[
H^*(XP(n)) = N(m) \oplus S
\]

where \(N(m)\) is a truncated polynomial algebra over \(k\) and \(S \cup H^*(XP(n)) = 0\). Theorem 2 considers the case \(k = Z(p)\) and states that

\[
H^*(XP(n)) = \tilde{N}(n) \oplus \tilde{S}
\]

where \(\tilde{N}\) is a truncated polynomial algebra on generators in even dimensions and \(\tilde{S}\) is an \(A(p)\)-subalgebra of \(H^*(XP(n))\) so that an \(A(p)\)-algebra structure can be induced on \(\tilde{N}\). In a subsequent paper Theorem 2 will be used to study the action of the Steenrod algebra, \(A(p)\), on \(H^*(XP(n), Z(p))\), and \(H^*(X, Z(p))\).

In [2] William Browder and Emery Thomas studied the \(Z(2)\)-cohomology of \(XP(2)\), and in [3] it was pointed out that Borel's methods can be used to obtain the \(Z(p)\)-cohomology of \(XP(\infty)\) when it exists (if and only if the space has an associative multiplication [10]). Steenrod and Rothenberg...
studied $H^*(XP(n))$ for associative $H$-spaces in [9]. Theorems 1 and 2 extend the results of [2], [3], and [9] by considering $XP(n)$, $n > 2$, for $H$-spaces other than topological groups.

1. Main results. We begin with some notation. If $V$ is a graded vector space over the field $k$, then $V^o$ and $V^e$ will denote the subspaces of odd and even dimensional elements respectively. The free commutative algebra generated by $V$ is

$$U(V) = \Lambda(V^o) \otimes k(V^e) \quad \text{if char } k \neq 2$$

and

$$U(V) = k(V) \quad \text{if char } k = 2.$$  

The exterior algebra is denoted by $\Lambda$ and $k(V)$ is the polynomial algebra. Let $U(V/t)$ be the truncated algebra of height $t$ generated by $V$.

An $A_n$-structure on $X$, [10], is a quasi-fibration

$$p_n: (E(n), E(n - 1), \ldots, X) \rightarrow (XP(n - 1), XP(n - 2), \ldots, *)$$

with fiber $X$. The space $E(m) = X \circ m \circ X$ is the $m$-fold join of $X$, [7], with the usual inclusion into $E(m + 1)$, and $XP(m) = c_{p_m}$ the mapping cone of $p_m$ which is $p_n$ restricted to $E(m)$. Let $T_m$ denote the vector space of primitive elements of $H^*(X)$ which are transgressive in the quasi-fibration

$$X \rightarrow E(m) \xrightarrow{p_m} XP(m - 1).$$

The set $x = \{x_i\}$ is a vectorspace basis for $T_m$. Let $y_i \in H^*(XP(m - 1))$ be a transgression of $x_i$ and $\otimes_{m} = \{y_i\}$. The mapping cone of $p_m$ is $XP(m)$ and there is the exact cohomology sequence

$$\ldots \rightarrow H^{n-1}(E(m)) \xrightarrow{\delta} H^n(XP(m)) \xrightarrow{j^*} H^n(XP(m - 1))$$

$\delta$ the inclusion of $XP(m - 1)$ into $XP(m)$. Since $p_m^*(y_i) = 0$, choose $z_i \in H^*(P(m, X))$ to be such that $j^*(z_i) = y_i$ and $\otimes = \{z_i\}$. The element $z_i$ will be called a $(m + 1)$-transgression of $x_i$. Set $N(m) = U(Z/m + 1)$.

**Theorem 1.** If $H^*(X)$ is primitively generated and $XP(m)$ is defined, then there exists a trivial $k$-algebra $S$ such that as $k$-algebras

$$H^*(XP(m)) \cong N(m) \oplus S.$$ 

More specific results are possible if $k = Z(p), p$ a prime. Let $J$ be the ideal of $N(m)$ generated by the odd dimensional elements of $N(m)$ and $\tilde{N}(m)$ the subalgebra generated by $\otimes$, then $N(m) = \tilde{N}(m) \oplus J$. Define $\hat{S} = S \oplus J$.

**Theorem 2.** If $H^*(X)$ is primitively generated, $k = Z(p)$, and $XP(m)$ is defined, then $\hat{S}$ is an $A(p)$-module and there is the vector space isomorphism

$$H^*(XP(m)) \cong \tilde{N}(m) \oplus \hat{S}$$

so that it is possible to induce an $A(p)$-algebra structure on $\tilde{N}(m)$.
2. Proof of Theorem 1. Theorem 1 is proved by induction starting with \( P(1, X) = SX \), the reduced suspension of \( X \). In [2] it was shown that the primitive elements are the \( 1 \)-transgressive elements of \( H^*(X) \) so that Theorem 1 is immediately satisfied for this case.

Induction hypothesis: \( H^*(XP(n - 1)) = N' \oplus S' \) as in the theorem, where \( N' \) is generated by \( \mathcal{G}' \subseteq H^*(XP(n - 1)) \), being constructed as \( \mathcal{L} \) was above. Since the \( n \)-transgressive elements are \((n - 1)\)-transgressive, we may choose \( \mathcal{G}' \supseteq \mathcal{G} \).

Let \( D \) be a subspace of \( H^*(X) \) complementary to \( T_1 = T \) and let \( \mathcal{Y} \) be a basis for \( T \). Since \( H^q(X) \) is finite dimensional as a vector space for all \( q \), we can choose a dual basis \( \mathcal{Y}' \) for \( T^* \subseteq H_*(X) \) such that if \( x_i \in \mathcal{Y} \) and \( w_j \in \mathcal{Y}' \), then \( \langle x_i, w_j \rangle \) is 1 if \( i = j \) and 0 otherwise.

**Lemma 2.1.** There is an isomorphism \( f: \bigoplus H^*(X) \to H^*(E(n)) \) such that if each \( x_i \) is primitive and transgresses to \( z_i \), then

\[
\delta (f(x_1 \otimes \cdots \otimes x)) = \delta (x_1 \ast \cdots \ast x_n) = \pm z_1 \cup \cdots \cup z_n.
\]

The existence of an isomorphism is known from [7]. The formula is obtained from a direct application of Theorem (2.4) of [15].

Letting \( f' \) be the dual of \( f \), \( f': H_*(E(n)) \to \bigotimes H_*(X) \) is also an isomorphism. Now define \( S' = f(S_1) \) where \( S_2 = (T \otimes D) + (D \otimes T) + (D \otimes D) \) and \( S_{i+1} = (T + D) \otimes S_i \). We then define \( S = \delta(S') \).

We now show that \( S \cap N = 0 \) and that the products of less than \( n + 1 \) elements of \( Z \) are linearly independent. Let \( z \in S \cap N \). Since \( z \in N \), \( z = \Sigma a_{i,j} z_{i,j} \) where this denotes a finite sum, \( a_{i,j} \in k \), and

\[
\bar{z}_{i,j} = z_{i(1)} \cup \cdots \cup z_{i(j)}
\]

is the product of \( j \) elements of \( Z \). For convenience it is assumed that the cup product \( \bar{z}_{i,j} \) is taken in such a manner that the indices are nondecreasing from left to right. Since \( z \in X \), there is \( s \in S' \) such that \( \delta(s) = z \), and therefore, \( j^*(z) = 0 \). Observe that \( XP(n) \) is of category \( n + 1 \) so that \( \bar{z}_{i,m} = 0 \) for \( m > n \). Now \( j^*(z_i) = y_i \in \mathcal{G}' \) so \( j^*(z) = \Sigma a_{i,j} y_{i,j} \). By the induction assumption, \( a_{i,j} = 0 \) for \( j < n \); hence, \( z = \Sigma a_{i,j} \bar{z}_{i,j} \). By Lemma 2.1

\[
\bar{z}_{i,n} = \varepsilon_{i,n} \delta(x_{i,n}), \quad \varepsilon_{i,n} = \pm 1,
\]

where \( x_{i,n} = x_{i(1)} \ast \cdots \ast x_{i(n)} \). Let \( a = \Sigma \varepsilon_{i,n} a_{i,n} x_{i,n} \in H^*(E(n)) \) so that \( \delta(a) = \delta(s) = \pm 1 \), and \( \delta(a - s) = 0 \). Let \( c \in H^*(XP(n)) \) be such that \( p^*(c) = a - s \) and define

\[
w_{i,n} = f'^{-1}(w_{j(1)} \otimes \cdots \otimes w_{j(n)}) = w_{j(1)} \ast \cdots \ast w_{j(n)}.
\]

Note that \( \langle x_{i,n}, w_{i,n} \rangle \) is 1 if each \( i_k = j_k \) and is 0 otherwise. Let \( \alpha = \deg w_{i(2)} \) and \( \beta = \deg w_{i(2)} \). If \( x' = w_{i,n} - (-1)^{\beta} w_{i(2)} \ast w_{i(1)} \ast \cdots \ast w_{i(n)} \), then

\[
\langle a - s, x' \rangle = \langle a, x' \rangle - \langle s, x' \rangle = \langle a, w_{i,n} \rangle = a_{i,n}.
\]
Lemma 2.2. \( p_\ast(x') = 0 \).

Hence, \( \langle a - s, x' \rangle = \langle p^\ast(c), x' \rangle = \langle c, p_\ast(x') \rangle = 0 \) for \( i_1 \neq i_2 \) so that in this case \( a_{i,n} = 0 \). If \( \text{deg } z_{i(1)} \) is odd, \( p \neq 2 \) and \( i_1 = i_2 \), then

\[
z_{i(1)} \cup z_{i(2)} = -z_{i(1)} \cup z_{i(2)} = 0
\]

so that \( z_{i,n} = 0 \).

Lemma 2.3. Let \( i_1 = i_2 \). If \( p \neq 2 \) and \( \text{deg } w_{i(1)} \) is odd, or \( p = 2 \), then \( p^\ast(w_{i,1}) = 0 \).

Hence, \( a_{i,n} = 0 \) so that \( z = 0 \).

We next show that \( H^\ast(XP(n)) = N + S \). If \( x \in H^\ast(XP(n)) \), then \( j^\ast(z) \in N' \) so \( j^\ast(z) = \sum a_{i,j}z_{i,j} \). Let \( a = \sum a_{i,j}z_{i,j} \in N \), then \( j^\ast(z - a) = 0 \). Choose \( s \in H^\ast(E(n)) \) so that \( \delta(s) = z - a \). Now \( \delta(H^\ast(E(n))) \subseteq N + S \) and \( a \in N \) so \( z \in N + S \).

3. Proof of Theorem 2. Let \( T^0 \) be the linear subspace of \( H^\ast(X) \) generated by \( \mathcal{X} \) the odd dimensional elements of \( \mathcal{X} \) and define \( D^0 \) to be the linear subspace generated by \( D \) and \( \mathcal{X} \), the even dimensional elements of \( \mathcal{X} \). Define \( U_2 = (T^0 \otimes D^0) + (D^0 \otimes T^0) + (D^0 \otimes D^0) \) \( \otimes U_1 \). Let \( U' = f(U_1) \). If \( L^0 \) is the linear subspace of \( H^\ast(XP(n)) \) generated by \( \mathcal{X} \), then \( \hat{S} = L + \delta(U') \). Notice that since \( \hat{A}(p) \) is all in even degrees, \( \hat{A}(p)(T^0) \subseteq T^0 \). Now \( U + (T^0 \otimes \cdots \otimes T^0) = \bigotimes \hat{H}^\ast(X) \) and \( T^0 \otimes \cdots \otimes T^0 \) is an \( \hat{A}(p) \)-module. Hence, \( U \) has an \( \hat{A}(p) \)-module structure and consequently \( \delta(f(U)) \) is an \( A(p) \)-module. Since the elements of \( \tilde{N} \) are all of even degree and \( L \) is all in odd degrees, \( \hat{A}(p)L \cap N = \{0\} \). Hence, \( \hat{S} \) is an \( A(p) \)-module. By the Cartan formula, \( \hat{S} \) is an \( \hat{A}(p) \)-algebra.

4. Proof of lemmas. The proofs of Lemmas 2.1, 2.2, and 2.3 depend on the definitions of \( E(n) \) and \( XP(n) \) described in §1.

Proof of Lemma 2.1. Consider the Mayer-Vietoris sequence

\[
\cdots \to H^\ast(E(n)) \xrightarrow{i} H^\ast(E(n - 1)) \oplus H^\ast(X) \xrightarrow{j} H^\ast(E(n - 1) \times X) \xrightarrow{\delta} \cdots
\]

The map \( \delta \) is an epimorphism since \( i = 0 \). Since \( \text{Ker } \delta = (H^\ast(E(n - 1)) \otimes k) + (k \otimes H^\ast(X)) \subseteq H^\ast(E(n - 1) \Lambda X) \), it is immediate that

\[
f_n = \delta \eta^\ast: H^\ast(E(n - 1) \Lambda X) \to H^\ast(E(n))
\]

is an isomorphism where \( \eta: X \times X \to X \Lambda X \) is the quotient map. If for \( n = 2 \), \( f = f_2 \) and for \( n = k + 1 \), \( f = f_{k+1}(f_k \otimes 1) \), then \( f \) will be an isomorphism for all \( n \).

If \( w \in H^\ast(E(n - 1)) \) and \( x \in H^\ast(X) \), then \( w \star x = \delta(w \otimes x) \in H^\ast(E(n)) \) as defined in [13]. By the way \( f_n \) was constructed, \( f_n(w \otimes x) = w \star x \). Hence, we conclude that

\[
f(x_{i(1)} \otimes \cdots \otimes x_{i(n)}) = x_{i(1)} \ast \cdots \ast x_{i(n)}
\]

as required.
Proof of Lemma 2.2. For $n = 2$ we have the Hopf construction $p: X \circ X \to SX$. Let $(X \circ X, E, X')$ and $(SX, M, C)$ be the usual decompositions of these spaces, so that $p$ is a map of these triples. Recall that $p|X \times X = m$, the $H$-space multiplication. Now consider the diagram below

\begin{tikzcd}
H_*(X) \otimes H_*(X) & H_*(X \wedge X) & H_*(X) \otimes H_*(X) \\
H_*(X \times X) \arrow{d}{m} \arrow{r}{\delta} & H_*(X \circ X) \arrow{d}{p} \\
H_*(X) \arrow{r}{\sigma} & H_*(SX)
\end{tikzcd}

where $m_*$ is the Pontrjagin product. Let $w_{(1)}$ and $w_{(2)} \in \bar{H}_*(X)$, then since the diagram is commutative,

$$p_*(x') = \sigma(m_*(w_{(1)}, w_{(2)}) - (-1)^{\deg(w_{(1)}\deg(w_{(2)})} m_*(w_{(1)}, w_{(2)})).$$

Now since $H^*(X)$ is primitively generated, $H_*(X)$ has a commutative Pontrjagin product, [8], and $p_*(w) = 0$.

The proof can be completed by an induction argument using the fact that $p$ is a map of triples.

Proof of Lemma 2.3. This is an induction argument very much like the one above. Let $w = w_{(1)} = w_{(2)}$. By the proof of Lemma 2.2, $p_*(w \ast w) = w^2$. If the characteristic is 2, then $w^2 = 0$ since $H_*(X)$ has a commutative Pontrjagin product. If the characteristic is not 2, then $w$ is odd dimensional so that $w^2 = -w^2 = 0$. Using the decomposition of $E(n)$ and $P(n, X)$ mentioned above, an induction argument completes the proof of this lemma.

Bibliography


**DEPARTMENT OF MATHEMATICAL SCIENCES, VIRGINIA COMMONWEALTH UNIVERSITY, 901 W. FRANKLIN ST., RICHMOND, VIRGINIA 23284**

*Current address*: DOT/FAA/NAFEC, ANA-751, Bldg. 2, Atlantic City, New Jersey 08405