THE COHOMOLOGY OF THE PROJECTIVE n-PLANE

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Abstract. An H-space is a topological space with a continuous multiplication and an identity element. In this paper X has the homotopy type of a countable CW-complex with integral cohomology of finite type and primitively generated k-cohomology, k a field. The projective n-plane of X is denoted XP(n). The main results of this paper are: Theorem 1 which states that \( H^*(XP(n)) = N \oplus S \) where \( N \) is a truncated polynomial algebra over k and S is a trivial k-ideal, and Theorem 2 which considers the case \( k = \mathbb{Z}(p) \) and states that \( H^*(XP(n)) = \tilde{N} \oplus \tilde{S} \) where \( \tilde{N} \) is a truncated polynomial algebra on generators in even dimensions and \( \tilde{S} \) is an \( A(p) \)-subalgebra of \( H^*(XP(n)) \) so that an \( A(p) \)-algebra structure can be induced on \( \tilde{N} \). These theorems extend results by A. Borel, W. Browder, M. Rothenberg, N. E. Steenrod, and E. Thomas.

0. Introduction. An H-space is a topological space with a continuous multiplication and an identity element. In [10] Stasheff defined the projective n-plane of an H-space. In this paper X has the homotopy type of a countable CW-complex with integral cohomology of finite type and primitively generated k-cohomology, k a field. The two main results of this paper pertain to the cohomology of the projective n-plane, XP(n), of an H-space, X. Theorem 1 states that

\[ H^*(XP(n)) = N(m) \oplus S \]

where \( N(m) \) is a truncated polynomial algebra over k and \( S \cup H^*(XP(n)) = 0 \). Theorem 2 considers the case \( k = \mathbb{Z}(p) \) and states that

\[ H^*(XP(n)) = \tilde{N}(n) \oplus \tilde{S} \]

where \( \tilde{N} \) is a truncated polynomial algebra on generators in even dimensions and \( \tilde{S} \) is an \( A(p) \)-subalgebra of \( H^*(XP(n)) \) so that an \( A(p) \)-algebra structure can be induced on \( \tilde{N} \). In a subsequent paper Theorem 2 will be used to study the action of the Steenrod algebra, \( A(p) \), on \( H^*(XP(n), \mathbb{Z}(p)) \), and \( H^*(X, \mathbb{Z}(p)) \).

In [2] William Browder and Emery Thomas studied the \( \mathbb{Z}(2) \)-cohomology of \( XP(2) \), and in [3] it was pointed out that Borel's methods can be used to obtain the \( \mathbb{Z}(p) \)-cohomology of \( XP(\infty) \) when it exists (if and only if the space has an associative multiplication [10]). Steenrod and Rothenberg...
studied $H^*(XP(n))$ for associative $H$-spaces in [9]. Theorems 1 and 2 extend the results of [2], [3], and [9] by considering $XP(n)$, $n > 2$, for $H$-spaces other than topological groups.

1. **Main results.** We begin with some notation. If $V$ is a graded vector space over the field $k$, then $V^o$ and $V^e$ will denote the subspaces of odd and even dimensional elements respectively. The free commutative algebra generated by $V$ is $U(V) = \Lambda(V^o) \otimes k(V^e)$ if char $k \neq 2$
and
$U(V) = k(V)$ if char $k = 2$.
The exterior algebra is denoted by $\Lambda$ and $k(V)$ is the polynomial algebra. Let $U(V/t)$ be the truncated algebra of height $t$ generated by $V$.

An $A_n$-structure on $X$, [10], is a quasi-fibration

$$p_n: (E(n), E(n-1), \ldots, X) \rightarrow (XP(n-1), XP(n-2), \ldots, *)$$
with fiber $X$. The space $E(m) = X \circ m \circ X$ is the $m$-fold join of $X$, [7], with the usual inclusion into $E(m+1)$, and $XP(m) = c_m$ the mapping cone of $p_m$ which is $p_n$ restricted to $E(m)$. Let $T_m$ denote the vector space of primitive elements of $H^*(X)$ which are transgressive in the quasi-fibration

$$X \rightarrow E(m) \leftarrow XP(m - 1).$$
The set $x = \{x_i\}$ is a vectorspace basis for $T_m$. Let $y_i \in H^*(XP(m-1))$ be a transgression of $x_i$ and $\otimes_m = \{y_i\}$. The mapping cone of $p_m$ is $XP(m)$ and there is the exact cohomology sequence

$$\ldots \rightarrow H^{n-1}(E(m)) \xrightarrow{\delta} H^n(XP(m)) \rightarrow H^n(XP(m-1)) \xrightarrow{j^*} \ldots,$$

and $j$ the inclusion of $XP(m-1)$ into $XP(m)$. Since $p^*_m(y_i) = 0$, choose $z_i \in H^*(P(m, X))$ to be such that $j^*(z_i) = y_i$ and $\otimes = \{z_i\}$. The element $z_i$ will be called a $(m+1)$-transgression of $x_i$. Set $N(m) = U(Z/m + 1)$.

**Theorem 1.** If $H^*(X)$ is primitively generated and $XP(m)$ is defined, then there exists a trivial $k$-algebra $S$ such that as $k$-algebras

$$H^*(XP(m)) \approx N(m) \oplus S.$$

More specific results are possible if $k = Z(p)$, $p$ a prime. Let $J$ be the ideal of $N(m)$ generated by the odd dimensional elements of $N(m)$ and $\tilde{N}(m)$ the subalgebra generated by $\otimes$, then $N(m) = \tilde{N}(m) \oplus J$. Define $\hat{S} = S \oplus J$.

**Theorem 2.** If $H^*(X)$ is primitively generated, $k = Z(p)$, and $XP(m)$ is defined, then $\hat{S}$ is an $A(p)$-module and there is the vector space isomorphism

$$H^*(XP(m)) \approx \tilde{N}(m) \oplus \hat{S}$$
so that it is possible to induce an $A(p)$-algebra structure on $\tilde{N}(m)$.
2. Proof of Theorem 1. Theorem 1 is proved by induction starting with $P(1, X) = SX$, the reduced suspension of $X$. In [2] it was shown that the primitive elements are the 1-transgressive elements of $H^*(X)$ so that Theorem 1 is immediately satisfied for this case.

Induction hypothesis: $H^*(XP(n - 1)) = N' \oplus S'$ as in the theorem, where $N'$ is generated by $\mathfrak{N}' \subseteq H^*(XP(n - 1))$, being constructed as $\mathfrak{L}$ was above. Since the $n$-transgressive elements are $(n - 1)$-transgressive, we may choose $\mathfrak{N}' \supseteq \mathfrak{N}$.

Let $D$ be a subspace of $H^*(X)$ complementary to $T_1 = T$ and let $\mathfrak{X}$ be a basis for $T$. Since $H^q(X)$ is finite dimensional as a vector space for all $q$, we can choose a dual basis $\mathfrak{X}'$ for $T^* \subseteq H_*(X)$ such that if $x_i \in \mathfrak{X}$ and $w_j \in \mathfrak{X}'$, then $\langle x_i, w_j \rangle = 1$ if $i = j$ and 0 otherwise.

**Lemma 2.1.** There is an isomorphism $f: \bigoplus H^*(X) \to H^*(E(n))$ such that if each $x_i$ is primitive and transgresses to $z_i$, then

$$\delta(f(x_1 \otimes \cdots \otimes x_n)) = \delta(x_1 \ast \cdots \ast x_n) = \pm z_1 \cup \cdots \cup z_n.$$  

The existence of an isomorphism is known from [7]. The formula is obtained from a direct application of Theorem (2.4) of [15].

Letting $f^*$ be the dual of $f$, $f^*$: $\overline{H}_*(E(n)) \to \bigotimes H_*(X)$ is also an isomorphism. Now define $S' = f(S_2)$ where $S_2 = (T \otimes D) + (D \otimes T) + (D \otimes D)$ and $S_{j+1} = (T + D) \otimes S_j$. We then define $S = \delta(S')$.

We now show that $S \cap N = 0$ and that the products of less than $n + 1$ elements of $Z$ are linearly independent. Let $z \in S \cap N$. Since $z \in N$, $z = \Sigma a_{i,j} \tilde{z}_{i,j}$ where $a_{i,j} \in k$, and

$$\tilde{z}_{i,j} = z_{i(1)} \cup \cdots \cup z_{i(j)}$$

is the product of $j$ elements of $Z$. For convenience it is assumed that the cup product $\tilde{z}_{i,j}$ is taken in such a manner that the indices are nondecreasing from left to right. Since $z \in X$, there is $s \in S'$ such that $\delta(s) = z$, and therefore, $j^*(z) = 0$. Observe that $XP(n)$ is of category $n + 1$ so that $\tilde{z}_{i,m} = 0$ for $m > n$. Now $j^*(z) = y_i \in \mathfrak{Y}$ so $j^*(z) = \Sigma a_{i,j}y_{i,j}$. By the induction assumption, $a_{i,j} = 0$ for $j < n$; hence, $z = \Sigma a_{i,j} \tilde{z}_{i,j}$. By Lemma 2.1

$$\tilde{z}_{i,n} = \epsilon_{i,n} \delta(x_{i,n}), \quad \epsilon_{i,n} = \pm 1,$$

where $x_{i,n} = x_{i(1)} \ast \cdots \ast x_{i(n)}$. Let $a = \Sigma \epsilon_{i,n} a_{i,n} x_{i,n} \in H^*(E(n))$ so that $\delta(a) = \delta(s) = z$ and $\delta(a - s) = 0$. Let $c \in H^*(XP(n))$ be such that $p^*(c) = a - s$ and define

$$w_{j,n} = f^*^{-1}(w_{j(1)} \otimes \cdots \otimes w_{j(n)}) = w_{j(1)} \ast \cdots \ast w_{j(n)}.$$  

Note that $\langle x_{i,n}, w_{i,n} \rangle = 1$ is each $i_k = j_k$ and is 0 otherwise. Let $\alpha = \deg w_{i(2)}$ and $\beta = \deg w_{i(2)}$. If $x' = w_{j,n} - (-1)^n \beta w_{i(2)} \ast w_{j(1)} \ast \cdots \ast w_{j(n)}$, then

$$\langle a - s, x' \rangle = \langle a, x' \rangle - \langle s, x' \rangle = \langle a, w_{i,n} \rangle = a_{i,n}.$$
Lemma 2.2. $p_\ast(x') = 0$.

Hence, $\langle a-s, x' \rangle = \langle p_\ast(c), x' \rangle = \langle c, p_\ast(x') \rangle = 0$ for $i_1 \neq i_2$ so that in this case $a_i, n = 0$. If deg $z_{i(1)}$ is odd, $p \neq 2$ and $i_1 = i_2$, then

$$z_{i(1)} \cup z_{i(2)} = -z_{i(1)} \cup z_{i(2)} = 0$$

so that $z_{i, n} = 0$.

Lemma 2.3. Let $i_1 = i_2$. If $p \neq 2$ and deg $w_{i(1)}$ is odd, or $p = 2$, then $p_\ast(w_{i, j}) = 0$.

Hence, $a_i, n = 0$ so that $z = 0$.

We next show that $H^\ast(XP(n)) = N + S$. If $x \in H^\ast(XP(n))$, then $j_\ast(z) \in N'$ so $j_\ast(z) = \sum a_i, j u_{i, j}$. Let $a = \sum a_i, j z_{i, j} \in N$, then $j_\ast(z - a) = 0$. Choose $s \in H^\ast(E(n))$ so that $a = s - a$. Now $\delta(s) = z - a$. If for $n = 2, f = f_2$ and for $n = k + 1, f = f_k, 1(f_k \otimes 1)$, then $f$ will be an isomorphism for all $n$.

If $w \in H^\ast(E(n - 1))$ and $x \in H^\ast(X)$, then $w \otimes x = \delta(w \otimes X) \in H^\ast(E(n))$ as defined in [13]. By the way $f_n$ was constructed, $f_n(w \otimes x) = w \otimes x$. Hence, we conclude that

$$f(x_{i(1)} \otimes \cdots \otimes x_{i(n)}) = x_{i(1)} \ast \cdots \ast x_{i(n)}$$

as required.

3. Proof of Theorem 2. Let $T^0$ be the linear subspace of $H^\ast(X)$ generated by $X^\circ$ the odd dimensional elements of $X$ and define $D^0$ to be the linear subspace generated by $D$ and $X^\circ$, the even dimensional elements of $X$. Define $U_2 = (T^0 \otimes D^0) + (D^0 \otimes T^0) + (D^0 \otimes D^0)$ and $U_{r+1} = (T^0 + D^0) \otimes U_r$. Let $U' = f(U_n)$. If $L^0$ is the linear subspace of $H^\ast(XP(n))$ generated by $L^0$, then $\hat{S} = L + \delta(U')$. Notice that since $\hat{A}(p)$ is all in even degrees, $\hat{A}(p)(T^0) \subseteq T^0$. Now $U + (T^0 \otimes \cdots \otimes T^0) = \otimes X^\ast \hat{H}^\ast(X)$ and $T^0 \otimes \cdots \otimes T^0$ is an $\hat{A}(p)$-module. Hence, $U$ has an $\hat{A}(p)$-module structure and consequently $\delta(f(U))$ is an $A(p)$-module. Since the elements of $\hat{N}$ are all of even degree and $L$ is all in odd degrees, $\hat{A}(p)(N) \cap N = \{0\}$. Hence, $\hat{S}$ is an $A(p)$-module. By the Cartan formula, $\hat{S}$ is an $\hat{A}(p)$-algebra.

4. Proof of lemmas. The proofs of Lemmas 2.1, 2.2, and 2.3 depend on the definitions of $E(n)$ and $XP(n)$ described in §1.

Proof of Lemma 2.1. Consider the Mayer-Vietoris sequence

$$\cdots \rightarrow H^\ast(E(n)) \rightarrow H^\ast(E(n - 1)) \oplus H^\ast(X) \rightarrow H^\ast(E(n - 1) \times X) \rightarrow \delta \rightarrow \cdots$$

The map $\delta$ is an epimorphism since $i = 0$. Since $\text{Ker} \delta = (H^\ast(E(n - 1)) \otimes k) + (k \otimes H^\ast(X)) \subseteq H^\ast(E(n - 1) \Lambda X)$, it is immediate that

$$f_n = \delta \eta_\ast: H^\ast(E(n - 1) \Lambda X) \rightarrow H^\ast(E(n))$$

is an isomorphism where $\eta: X \times X \rightarrow X \Lambda X$ is the quotient map. If for $n = 2, f = f_2$ and for $n = k + 1$, $f = f_k, 1(f_k \otimes 1)$, then $f$ will be an isomorphism for all $n$.

If $w \in H^\ast(E(n - 1))$ and $x \in H^\ast(X)$, then $w \otimes x = \delta(w \otimes x) \in H^\ast(E(n))$ as defined in [13]. By the way $f_n$ was constructed, $f_n(w \otimes x) = w \otimes x$. Hence, we conclude that

$$f(x_{i(1)} \otimes \cdots \otimes x_{i(n)}) = x_{i(1)} \ast \cdots \ast x_{i(n)}$$

as required.
Proof of Lemma 2.2. For \( n = 2 \) we have the Hopf construction \( p: X \circ X \to SX \). Let \( (X \circ X, E, X') \) and \((SX, M, C)\) be the usual decompositions of these spaces, so that \( p \) is a map of these triples. Recall that \( p|X \times X = m \), the \( H \)-space multiplication. Now consider the diagram below

\[
\begin{array}{ccc}
H_*(X) \otimes H_*(X) \cong H_*(X \wedge X) & \leftrightarrow & H_*(X) \otimes H_*(X) \\
\eta_* & \downarrow m_* & \downarrow f^{-1} \\
H_*(X \times X) & \delta & H_*(X \circ X) \\
\downarrow m_* & \downarrow \sigma & \downarrow p_* \\
H_*(X) & & H_*(SX)
\end{array}
\]

where \( m_* \) is the Pontrjagin product. Let \( w_{(1)} \) and \( w_{(2)} \) \( \in \bar{H}_*(X) \), then since the diagram is commutative,

\[
p_*(x') = \sigma(m_*(w_{(1)}), w_{(2)}) - (-1)^{\deg(w_{(1)})\deg(w_{(2)})} m_*(w_{(1)}, w_{(2)}).
\]

Now since \( H^*(X) \) is primitively generated, \( H_*(X) \) has a commutative Pontrjagin product, [8], and \( p_*(w) = 0 \).

The proof can be completed by an induction argument using the fact that \( p \) is a map of triples.

Proof of Lemma 2.3. This is an induction argument very much like the one above. Let \( w = w_{(1)} = w_{(2)} \). By the proof of Lemma 2.2, \( p_*(w \circ w) = w^2 \). If the characteristic is 2, then \( w^2 = 0 \) since \( H_*(X) \) has a commutative Pontrjagin product. If the characteristic is not 2, then \( w \) is odd dimensional so that \( w^2 = -w^2 = 0 \). Using the decomposition of \( E(n) \) and \( P(n, X) \) mentioned above, an induction argument completes the proof of this lemma.

Bibliography

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