SHORTER NOTES

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ON A COMPLETENESS THEOREM OF PALEY AND WIENER

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Abstract. In this brief note we offer a simplified proof of a classical completeness theorem for systems of complex exponential functions \( e^{i\lambda t} \).

A sequence of complex exponential functions \( \{ e^{i\lambda_n t} \}_{n=-\infty}^{\infty} \) is said to be exact in \( L^2[-\pi, \pi] \) if it is complete, that is, if the relations

\[
\phi \in L^2[-\pi, \pi] \quad \text{and} \quad \int_{-\pi}^{\pi} \phi(t) e^{i\lambda_n t} \, dt = 0 \quad (-\infty < n < \infty)
\]

imply that \( \phi(t) = 0 \) almost everywhere on \([ -\pi, \pi] \), but becomes incomplete upon the removal of a single term.

The following classical theorem of Paley and Wiener [2, p. 89] is fundamental. We offer a simplified version of the proof.

Theorem. Let \( \{ \lambda_n \}_{n=-\infty}^{\infty} \) be a symmetric sequence of real or complex numbers: \( \lambda_{-n} = -\lambda_n \) (\( n = 0, 1, 2, \ldots \)). If the system \( \{ e^{i\lambda_n t} \}_{n=-\infty}^{\infty} \) is exact in \( L^2[-\pi, \pi] \), then the infinite product

\[
f(z) = \prod_{n=1}^{\infty} (1 - z^2/\lambda_n^2)
\]

converges to an entire function of exponential type \( \pi \) for which

\[
\int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty \quad \text{and} \quad \int_{-\pi}^{\pi} |xf(x)|^2 \, dx = \infty.
\]

Proof. Since \( \{ e^{i\lambda_n t} \} \) is exact, there exists a function \( \phi \) in \( L^2[-\pi, \pi] \) for which

\[
\int_{-\pi}^{\pi} \phi(t) e^{i\lambda_n t} \, dt = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}
\]
Put

\[ g(z) = \int_{-\pi}^{\pi} \phi(t)e^{itz} dt. \]

Then \( g(z) \) is an entire function of exponential type \( \pi \), square integrable on the real axis, and zero at every \( \lambda_n \) (\( n \neq 0 \)). Assertion: \( g = f \). Observe first that \( g(z) \) can vanish only at the \( \lambda_n \)'s (\( n \neq 0 \)). Indeed, suppose that \( g(z) \) were zero for some other value, \( z = \gamma \) say. It is a simple matter to show that

\[ \frac{zg(z)}{z - \gamma} = \int_{-\pi}^{\pi} \phi_1(t)e^{itz} dt \]

for a suitable function \( \phi_1 \) in \( L^2([-\pi, \pi]) \) (see, for example, [1, p. 10]). Since the left side vanishes at every \( \lambda_n \), it follows that the system \( \{e^{it\lambda_n}\} \) is incomplete in \( L^2([-\pi, \pi]) \), contrary to assumption. Accordingly, \( g(z) \) has no zeros other than the \( \lambda_n \)'s (\( n \neq 0 \)), and hence we can write

\[ g(z) = e^{itz} \prod_{n=1}^{\infty} \left(1 - z^2/\lambda_n^2\right), \]

by virtue of Hadamard's factorization theorem. Since \( \{\lambda_n\} \) is symmetric, (1) holds with \( \phi(t) \) replaced by \( \phi(-t) \). But \( \phi(t) \) is uniquely determined, and so must be even. Therefore, \( g(z) \) is even, so \( A = 0 \) and hence \( g = f \).

For the second assertion, suppose to the contrary that \( xf(x) \) does belong to \( L^2 \) along the real axis. By the Paley-Wiener representation,

\[ zf(z) = \int_{-\pi}^{\pi} \mu(t)e^{itz} dt, \]

where \( \mu(t) \) is a nontrivial function in \( L^2([-\pi, \pi]) \). But \( zf(z) \) vanishes at every \( \lambda_n \). Once again, this contradicts the assumption that \( \{e^{it\lambda_n}\} \) is complete, and the result follows.

Remark. In [2] it is postulated that the \( \lambda_n \) have density 1 in each half-plane, \( x < 0 \) and \( x > 0 \). The above proof avoids this constraint. That such a density must in fact exist whenever \( \{e^{it\lambda_n}\} \) is exact is a consequence of a well-known property of entire functions (see, for example, [1, p. 25]).

References


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