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TRIVIAL EXTENSION OF A RING WITH BALANCED CONDITION

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ABSTRACT. A ring R is called QF-1 if every faithful R-module is balanced. In this paper we study commutative QF-1 rings. It is shown that a commutative QF-1 ring is local if and only if it is uniform. It is well known that commutative artinian QF-1 rings are QF, but Osofsky has constructed a nonartinian nonnoetherian commutative injective cogenerator, so QF-1, ring which is a trivial extension of a valuation ring. It is shown that if a trivial extension of a valuation ring is QF-1, then it has a nonzero socle. Furthermore such rings become injective cogenerator rings under certain conditions.

Throughout this paper rings are commutative rings with unity and modules are unital. Hence the trivial extension of a ring B by a B-module E is defined to be the ring whose additive group is the direct sum $B \oplus E$ with multiplication given by

$$(b, e) \cdot (b', e') = (bb', b'e + be').$$

An R-module M is called balanced if the canonical ring homomorphism of R into the double centralizer of M is surjective, and a ring R is said to be QF-1 if every faithful R-module is balanced. A ring R is said to be PF if R is an injective cogenerator as an R-module.

There is an interesting theorem on QF-1 rings that a commutative QF-1 artinian ring R is QF (V. P. Camillo [**1**] and S. E. Dickson and K. R. Fuller [**3**]) which has been generalized by C. M. Ringel [**10**], H. H. Storrer [**12**] and H. Tachikawa [**14**] under the weaker condition that R is noetherian or perfect instead of the assumption on R to be artinian.

On the other hand B. L. Osofsky [**9**, Example 1] has constructed a commutative nonnoetherian PF ring which is a trivial extension of a valuation ring. It is known that PF rings are QF-1, so her example distinguishes commutative QF-1 rings from QF rings. Hence trivial extension QF-1 rings of valuation rings are worthy to be considered. The purpose of this paper is to show that such a ring has a nonzero socle, which is an important necessary condition for a ring to be PF, and that it is PF under certain conditions.

V. P. Camillo [**2**] and H. H. Storrer [**11**] have proved that a commutative QF-1 ring R has the principal extension property which plays the important

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role in this paper, that is, every homomorphism of any principal ideal of R into R can be extended to all of R. Furthermore M. Ikeda and T. Nakayama [7] have proved that if a ring R has the principal extension property, then every principal ideal of R satisfies the annihilator condition, that is,

$$\mathrm{Ann}_R\, \mathrm{Ann}_R(Rr) = Rr \quad \text{for all } r \text{ in } R.$$

Although C. M. Ringel has proved in [**10**] that if a commutative QF-1 ring has a nonzero socle and if it is local then it is uniform, we shall prove it by using the principal extension property without assuming the existence of a nonzero socle and prove the converse of it.

THEOREM 1. *A commutative QF-1 ring is local if and only if it is uniform.*

PROOF. At first we shall consider the case where a local commutative QF-1 ring R has a nontrivial socle. Let S be a minimal ideal. For an arbitrary nonzero element x in R, an R-module $Rx/(\mathrm{Rad}(R))x$ is simple and isomorphic to S. Thus there exists a nonzero R-homomorphism of Rx into R whose image is S. The principal extension property means $S \subset Rx$. So R is uniform since S is essential in R.

Next we shall consider a local commutative QF-1 ring R with zero socle. In this case it is to be noted that the maximal ideal W $(= \mathrm{Rad}(R))$ of R is faithful. We shall prove the fact that if x and y are nonzero elements of R with $Rx \cap Ry = 0$, then $\mathrm{Ann}_R(x) + \mathrm{Ann}_R(y) = R$. This implies either $\mathrm{Ann}_R(x) = R$ or $\mathrm{Ann}_R(y) = R$, because R is assumed to be local. Hence either x or y is zero, which contradicts the assumption that x and y are nonzero elements. So suppose that $\mathrm{Ann}_R(x) + \mathrm{Ann}_R(y) \neq R$. Then W contains $\mathrm{Ann}_R(x) + \mathrm{Ann}_R(y)$, and an R-module $R/Wx \oplus R/Wy$ is faithful and so balanced, since $Wx \cap Wy \subset Rx \cap Ry = 0$. We can define a nonzero map ϕ of $R/Wx \oplus R/Wy$ into $R/Wx \oplus R/Wy$ as follows $(a, b) \mapsto (xa, yb)$ for $a \in R/Wx$, $b \in R/Wy$, since $Rx \neq Wx$ and $Ry \neq Wy$. Every element in $\mathrm{Hom}_R(R/Wx, R/Wy)$ is induced by some element r of R such that $Wxr \subset Wy$, so r annihilates x since $Wx \cap Wy = 0$ and W is faithful. Also since $Wyx \subset Wy \cap Wx = 0$, we have $\mathrm{Ann}_R(x) \supset Wy$. Consequently we have

$$T(R/Wx, R/Wy) = \mathrm{Ann}_R(x)/Wy,$$

where

$$T(R/Wx, R/Wy) = \sum \{\mathrm{Im}(f); f \in \mathrm{Hom}_R(R/Wx, R/Wy)\}.$$

Thus $(x - y)T(R/Wx, R/Wy) = 0$ since $\mathrm{Ann}_R(x) \subset W$. Similarly $(y - x)T(R/Wy, R/Wx) = 0$. Then by the Camillo criterion [**1**, 11. Lemma], ϕ is an element of the double centralizer of $R/Wx \oplus R/Wy$. So it should be induced by the multiplication with an element r of R. But then the element $(1 + Wx, 1 + Wy)$ in $R/Wx \oplus R/Wy$ is mapped onto $(r + Wx, r + Wy) = (x + Wx, y + Wy)$, thus $r - x \in Wx$ and $r - y \in Wy$, and therefore $r \in Rx \cap Ry = 0$, a contradiction.

Conversely, suppose that a commutative QF-1 ring R is uniform. We fix an arbitrary nonunit element x of R. For any element r of R, the element rx is a

nonunit. If $\mathrm{Ann}_R(rx) = 0$, a map of Rrx into R as follows $r'rx \mapsto r'$ for $r' \in R$ is a well-defined R-homomorphism. By the principal extension property, there is an element $p \in R$ such that $prx = 1$, which contradicts the fact that rx is a nonunit. Thus we have $\mathrm{Ann}_R(rx) \neq 0$. Since R is uniform and $\mathrm{Ann}_R(rx) \cap \mathrm{Ann}_R(1 - rx) = 0$, we have $\mathrm{Ann}_R(1 - rx) = 0$, so $1 - rx$ is a unit by the above argument. We have showed that every nonunit element is in the radical of R, which implies that R is local.

V. P. Camillo has proved in [**1**, Lemma 2] that if the direct sum of a faithful R-module M and a simple R-module S is balanced then either $\mathrm{Hom}_R(S, M)$ or $\mathrm{Hom}_R(M, S)$ is not zero. Hence it follows that if R is QF-1 and if a faithful R-module M has a zero socle, then the (Jacobson) radical $\mathrm{Rad}(M)$ of M is not equal to M. We note that for a module M over a local ring R, $\mathrm{Rad}(M)$ is equal to $\mathrm{Rad}(R) \cdot M$.

We need some definitions. A module E is called uniserial if the lattice of submodules of E is linearly ordered by inclusion, and a ring B is called a valuation ring if B is uniserial as a B-module. A valuation ring B is said to be maximal if every system of pairwise solvable congruences of the form

$$x \equiv x_\alpha(I_\alpha) \qquad (\alpha \in A, x_\alpha \in B, I_\alpha \text{ an ideal of } B)$$

has a simultaneous solution in B. We say B is almost maximal if the above congruences have a simultaneous solution whenever $\bigcap_{\alpha \in A} I_\alpha \neq 0$ (cf. C. Faith [**4**]).

Now, we are ready to prove our next theorem.

THEOREM 2. *Let R be the trivial extension ring of a valuation ring B by a nonzero B-module E. If R is QF-1, then the following hold*:

(a) *E is faithful and uniserial*;

(b) *the socle of R is equal to* $(0, \mathrm{Soc}_B(E))$ *and is not zero.*

PROOF. (a) R is a commutative local ring with the maximal ideal $(\mathrm{Rad}(B), E)$, and R is uniform by Theorem 1. $(\mathrm{Ann}_B(E), 0)$ and $(0, E)$ are ideals with zero intersection and so, by the uniformness of R, $\mathrm{Ann}_B(E)$ must be zero, that is, E is faithful.

For the second assertion it is sufficient to show that for any two elements x, y of E, it holds that either $Bx \subset By$ or $Bx \supset By$. Since B is a valuation ring, we may assume $\mathrm{Ann}_B(y)$ contains $\mathrm{Ann}_B(x)$. Then

$$\mathrm{Ann}_R(0, x) = (\mathrm{Ann}_B(x), E) \subset (\mathrm{Ann}_B(y), E) = \mathrm{Ann}_R(0, y).$$

Two principal ideals $R(0, x) = (0, Bx)$ and $R(0, y) = (0, By)$ satisfy the annihilator condition, so we have

$$(0, Bx) = \mathrm{Ann}_R\mathrm{Ann}_R(0, x) \supset \mathrm{Ann}_R\mathrm{Ann}_R(0, y) = (0, By),$$

which shows that $Bx \supset By$.

(b) Assume that there exists an element (b, e) of R which generates a minimal ideal such that $b \neq 0$. Noting that $bE \neq 0$, we have $R(b, e) \supsetneqq (0, bE) \neq 0$, which contradicts the minimality of $R(b, e)$. Thus $\mathrm{Soc}_R(R)$ must be $(0, \mathrm{Soc}_B(E))$.

Next we must show that R has a nonzero socle. If $\mathrm{Soc}_B(B)$ is not zero, then we have a minimal ideal $(0, Bbe)$, where b is a generator of a minimal ideal of B and e is an element of E such that $be \neq 0$.

Thus we may assume $\mathrm{Soc}_B(B) = 0$. Let W be the radical of B. It is to be noted that W is a faithful ideal of B.

We claim that E is not a cyclic B-module. Suppose E is cyclic with a generator e, then the ideal (W, We) of R is faithful and has a zero socle. Then we have $(W, We) \neq \mathrm{Rad}(W, We) = (W^2, We)$. Hence W is generated by one element w since B is a valuation ring. Then

$$(0, We) = \mathrm{Ann}_R\, \mathrm{Ann}_R(0, we) = \mathrm{Ann}_R(0, E) = (0, E),$$

which contradicts $E \neq We$.

Next, we claim that any proper submodule F of E is not faithful. We can take two elements e_1, e_2 of E so that

$$F \subset Be_1 \subsetneqq Be_2 \subset E.$$

From the proof of (a), $\mathrm{Ann}_B(e_1)$ contains strictly $\mathrm{Ann}_B(e_2)$; specifically $\mathrm{Ann}_B(e_1)$ is nonzero, so is $\mathrm{Ann}_B(F)$.

Now assume $\mathrm{Soc}_R(R) = 0$. Then the radical (W, E) of R is faithful and has a zero socle, so that $(W, E) \neq \mathrm{Rad}(W, E) = (W^2, WE)$. WE is a faithful B-module, so it is equal to E. Thus W is generated by one element w. $\mathrm{Rad}(R) = R(w, 0)$ leads to R having a nonzero socle by applying the proof of C. M. Ringel [**10**, Lemma 3], a contradiction. This completes the proof.

COROLLARY 3. *Let R be the trivial extension ring of a valuation ring B of an injective nonzero B-module E. If R is QF-1, then the following hold.*

(a) *E is the injective hull of $B/\mathrm{Rad}(B)$, so B is an almost maximal valuation ring;*

(b) *if E is cyclic, then B and R are PF;*

(c) *if B is not an integral domain, then R is PF.*

PROOF. (a) We know that $(0, \mathrm{Soc}_B(E))$ is minimal and essential in R from Theorems 1 and 2. So $\mathrm{Soc}_B(E)$ is simple and essential in E, and thus E is the injective hull of $B/\mathrm{Rad}(B)$. By C. Faith [**4**, Theorem 20.49], B is almost maximal.

(b) If E is cyclic, E is isomorphic to B. By (a), E is an injective cogenerator, so is B. By B. J. Müller [**8**, Theorem 10], then R is PF.

(c) Since B is an almost maximal nonintegral domain, B is maximal by C. Faith [**4**, Proposition 20.46]. We shall show that the endomorphism ring of E is canonically isomorphic to B. This implies that R is injective by R. M. Fossum et al. [**6**, Corollary 4.37] since E is injective. Now let f be any element of the endomorphism ring of E, and $\{e_\alpha\}_{\alpha \in A}$ be a set of generators of E. For every e_α, a map of $R(0, e_\alpha)$ into R, as follows $r(0, e_\alpha) \mapsto (0, bfe_\alpha)$ for $r = (b, e) \in R$, is a well-defined R-homomorphism, so there exists an element b_α of B such that $fe_\alpha = b_\alpha e_\alpha$ by the principal extension property. Then we consider

the system of congruences as follows:

$$x \equiv b_\alpha(I_\alpha) \qquad (\alpha \in A, b_\alpha \in B, I_\alpha = \mathrm{Ann}_B(e_\alpha)).$$

For any α and β in A, we may assume that $Be_\alpha \subset Be_\beta$ since E is uniserial. If $e_\alpha = be_\beta$ by an appropriate element b of B, then $b_\beta e_\alpha = bb_\beta e_\beta = bfe_\beta = fe_\alpha = b_\alpha e_\alpha$, so $b_\beta - b_\alpha \in I_\alpha$. This shows that the above system is pairwise solvable. There exists a solution of it, since B is maximal. This solution induces f. This completes the proof.

C. Faith [**5**, Theorem 6A] has given equivalent conditions on a trivial extension of a ring to be a PF valuation ring. Here we shall give the necessary and sufficient condition in order that a trivial extension QF-1 ring is a valuation ring.

COROLLARY 4. *Let R be the trivial extension QF-1 ring of a ring B by a nonzero B-module E. Then the following are equivalent*:

(a) *R is a valuation ring*;

(b) *B is an integral domain and is a valuation ring.*

PROOF. We assume (a). For any ideals I, J of B, (I, IE), (J, JE) are ideals of R. We have either $(I, IE) \supset (J, JE)$ or $(I, IE) \subset (J, JE)$, so either $I \supset J$ or $I \subset J$, which means that B is a valuation ring.

Then E is faithful by Theorem 2. Next, take any nonzero element b of B. An ideal of R generated by $(b, 0)$ is (Bb, bE), which is not contained in an ideal $(0, E)$ of R. Thus it contains $(0, E)$, hence we have $bE = E$. The regularity of b follows from the faithfulness of E.

We assume (b). By Theorem 2, E is uniserial and faithful. We are in the case where $\mathrm{Soc}_B(B) = 0$ because B is an integral domain. From the proof of Theorem 2, any proper submodule of E is not faithful. Take an ideal I of R such that I contains one element (b, e) with $b \neq 0$. Then a submodule bE of E is faithful, so equal to E, which leads to I containing $(0, E)$. Thus the ideals of R are of the form (J, E), with J an ideal of B, and of the form $(0, F)$, with F a submodule of E. Since both B and E are uniserial, R is a valuation ring.

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REFERENCES

1. V. P. Camillo, *Balanced rings and a problem of Thrall*, Trans. Amer. Math. Soc. **149** (1970), 143–153. MR **41** #5417.

2. _____, *A property of QF-1 rings* (preprint).

3. S. E. Dickson and K. R. Fuller, *Commutative QF-1 artinian rings are QF*, Proc. Amer. Math. Soc. **24** (1970), 667–670. MR **40** #5646.

4. C. Faith, *Algebra*. II. Ring Theory, Die Grundlehren der Math. Wissenchaften, no. 191, Springer-Verlag, Berlin and New York, 1976. MR **55** #383.

5. C. Faith, *Self-injective rings* (preprint).

6. R. M. Fossum, P. A. Griffith and I. Reiten, *Trivial extensions of abelian categories*, Lecture Notes in Math., vol. 456, Springer-Verlag, Berlin and New York, 1975. MR **52** #10810.

7. M. Ikeda and T. Nakayama, *On some characteristic properties of quasi-Frobenius and regular rings*, Proc. Amer. Math. Soc. **5** (1954), 15–19. MR **15**, 677.

8. B. J. Müller, *On Morita duality*, Canad. J. Math. **21** (1969), 1338–1347. MR **41** #258.

9. B. L. Osofsky, *A generalization of QF rings*, J. Algebra **4** (1966), 373–387; erratum, ibid. **9** (1968), 120. MR **34** #4305; MR **36** #6443.

10. C. M. Ringel, *Commutative QF-1 rings*, Proc. Amer. Math. Soc. **42** (1974), 365–368. MR **49** #9022.

11. H. H. Storrer, *Epimorphismen von kommutativen Ringen*, Comment. Math. Helv. **43** (1968), 378–401. MR **39** #4137.

12. _____, *Epimorphic extensions of noncommutative rings*, Comment. Math. Helv. **48** (1973), 72–86. MR **48** #342.

13. H. Tachikawa, *Quasi-Frobenius rings and generalizations*, Lecture Notes in Math., vol. 351, Springer-Verlag, Berlin and New York, 1973. MR **50** #2233.

14. _____, *Commutative perfect QF-1 rings*, Proc. Amer. Math. Soc. **68** (1978), 261–264.

15. R. M. Thrall, *Some generalizations of QF-algebras*, Trans. Amer. Math. Soc. **64** (1948), 173–183. MR **10**, 98.

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