THE LYNDON-HOCHSCHILD-SERRE SPECTRAL SEQUENCES FOR SHEAVES WITH OPERATORS

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ABSTRACT. Two spectral sequences associated with a $G$-sheaf and a normal subgroup of $G$ are given together with an application.

If $H$ is a normal subgroup of a group $G$, the Lyndon-Hochschild-Serre spectral sequence relates the cohomology of $H$ and of $G/H$ to that of $G$. In this note we give two analogous spectral sequences in the cohomology theory of groups with coefficients in sheaves with operators. Applications include a direct proof of a theorem of Conner and Raymond [1].

1. Sheaves with operators. We start by briefly reviewing the cohomology of groups with coefficients in sheaves with operators. For details, we refer to [1], [2], and [3]. Let $(G, X)$ be a continuous action of a group $G$ on a topological space $X$. A $G$-sheaf over $X$ is a sheaf $S$ of abelian groups over $X$ with an action $(G, S)$ such that the projection $S \to X$ is equivariant and that for each $g \in G$ and $x \in X$, $g: S_x \to S_{gx}$ is a group homomorphism between the stalks. The categories of $G$-sheaves over $X$ and of abelian groups are denoted, respectively, by $\mathcal{G}_X$ and $\mathcal{C}$. The functor $\Gamma^G_X: \mathcal{G}_X \to \mathcal{C}$ sends a $G$-sheaf $S$ to the group $\Gamma^G_X(S)$ of $G$-invariant sections of $S$ over $X$. For a $G$-sheaf $S$ over $X$, the cohomology $H^*(G, S)$ of $G$ with coefficients in $S$ is defined by $H^*(G, S) = R\Gamma^G_X(S)$, where $R\Gamma^G_X$ denotes the right derived functor of $\Gamma^G_X$.

2. The Lyndon-Hochschild-Serre spectral sequences for $G$-sheaves. Let $H$ be a normal subgroup of $G$ and let $\phi: X \to X/H$ be the canonical projection onto the orbit space. The category $\mathcal{G}_X$ can be thought of as a subcategory of $\mathcal{G}_X^H$ in a natural manner. If $M$ is a $G$-module, then the group $\Gamma^H(M) = M^H$ of $H$-invariant elements of $H$ becomes naturally a $G/H$-module. Thus the functor $\Gamma^H_X: \mathcal{G}_X^H \to \mathcal{C}$ sends $\mathcal{G}_X^H$ into the category $\mathcal{C}^{G/H}$ of $G/H$-modules, since it is factorized as $\Gamma^H_X = \Gamma^H \circ \Gamma_X$. We denote by $\mathcal{G}_X/H$ the category of sheaves of abelian groups over $X/H$. The functor $\phi^H_*: \mathcal{G}_X^H \to \mathcal{G}_X/H$ assigns, by definition, to each $H$-sheaf $S$ the sheaf over $X/H$ determined by the presheaf $U \mapsto \Gamma(\phi^{-1}(U), S)^H$. The restriction of $\phi^H_*$ to the subcategory $\mathcal{G}$

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can be thought of as a functor into the category $\mathcal{C}_{G/H}$ of $G/H$-sheaves over $X/H$ in a natural manner.

**Lemma.** If $S$ is an injective $G$-sheaf over $X$, then

1. $S$ is an injective $H$-sheaf over $X$,
2. $\Gamma^H_X(S)$ is an injective $G/H$-module,
3. $\varphi^H_*(S)$ is a $\Gamma^G_{X/H}$-acyclic $G/H$-sheaf over $X/H$, i.e.

$$R^n\Gamma^G_{X/H}(\varphi^H_*(S)) = H^n(G/H, \varphi^H_*(S)) = 0, \text{ for } n > 1.$$  

**Proof.** (1) Set $O = \mathcal{O}(H)$ in [2, Lemma 5.6.2]. (2) Recall the factorization $\Gamma^H_X = \Gamma^H \circ \Gamma_X$. By [2, Corollaire de Proposition 5.1.3], $\Gamma_X$ sends an injective $G$-sheaf to an injective $G$-module. On the other hand, $\Gamma^H$ sends an injective $G$-module to an injective $G/H$-module. (3) By (1) above and [2, Corollaire de Proposition 5.1.3], the sheaf $\varphi^H_*(S)$ is a flabby sheaf over $X/H$. Consider the spectral sequence [3, (1.2)] for $S = \varphi^H_*(S)$:

$$E^{p,q}_2 = H^p(G/H, H^q(X/H, \varphi^H_*(S))) \Rightarrow H^n(G/H, \varphi^H_*(S)).$$

Since $\varphi^H_*(S)$ is flabby, the spectral sequence degenerates to yield

$$H^n(G/H, \varphi^H_*(S)) \cong H^n(G/H, \varphi^0(X/H, \varphi^H_*(S)))$$

$$= H^n(G/H, \varphi^0(X/H, S)).$$

Also consider the Lyndon-Hochschild-Serre spectral sequence for the $G$-module $H^0(X, S)$:

$$E^{p,q}_2 = H^p(G/H, H^q(H, H^0(X, S))) \Rightarrow H^n(G, S).$$

By (1) above and [2, Corollaire de Proposition 5.1.3], $H^0(X, S) = \Gamma_X(S)$ is injective either as $G$-module or as $H$-module. Hence $H^n(H, H^0(X, S)) = 0 = H^n(G/H, H^0(X, S))$ for $n > 1$. Therefore we have $H^n(G/H, H^0(X, S)) = H^n(G, H^0(X, S)) = 0$, for $n > 1$. Q.E.D.

**Theorem.** If $S$ is a $G$-sheaf over $X$ and if $H$ is a normal subgroup of $G$, then for each nonnegative integer $q$, the group $H^q(H, S)$ has a canonical structure of $G/H$-module and the sheaf $R^q\varphi^H_*(S)$ that of $G/H$-sheaf over $X/H$ and there are two spectral sequences

$$\text{I}_E^{p,q} = H^p(G/H, R^q\varphi^H_*(S)) \Rightarrow H^n(G, S),$$

$$\text{II}_E^{p,q} = H^p(G/H, H^q(H, S)) \Rightarrow H^n(G, S).$$

**Proof.** Lemma (1) shows that $R\Gamma^H_X(S)$ and $\varphi^H_*(S)$ can be computed by taking a $G$-injective resolution of $S$. Thus, for each $q$, $H^q(H, S) = R^q\Gamma^H_X(S)$ has a canonical structure of $G/H$-module and $R^q\varphi^H_*(S)$ that of $G/H$-sheaf over $X/H$. With the aid of the lemma, [2, Théorème 2.4.1] is applied to the
commutative diagram

\[
\begin{array}{ccc}
\Gamma_X^H & \xrightarrow{\varphi_X^H} & \Gamma_X^G \\
\downarrow & & \downarrow \\
\Gamma_X^{G/H} & \xrightarrow{\mu_X^{G/H}} & \Gamma_X^{G/(H/H)}
\end{array}
\]

to obtain the spectral sequences.

3. **An application.** (3.10) Theorem in [1] is obtained from the above theorem as follows. Let \((N, W)\) be a properly discontinuous action of a group \(N\) on \(W\). A representation \(\Phi: N \to \text{GL}(2k, \mathbb{Z})\) defines an \(N\)-sheaf structure on the constant sheaf \(\mathcal{G}^{2k} = W \times \mathbb{Z}^{2k}\). Let \(L \subset N\) be a normal subgroup satisfying

(i) \(L\) acts freely on \(W\),

(ii) \(L \subset \text{Ker } \Phi\).

Denote by \(\mu: W \to W/L = B\) the canonical projection onto the orbit space. By (ii), \(\Phi\) defines an \(N/L\)-sheaf structure on the constant sheaf \(\mathcal{G}^{2k} = B \times \mathbb{Z}^{2k}\) on \(B\). We set \(X = W, G = N, H = L, \varphi = \mu\) and \(S = \mathcal{G}^{2k}\) in the first spectral sequence of the theorem. The condition (i) implies that \(R^q\mu_*^{G/H}(\mathcal{G}^{2k}) = 0\), for \(q > 1\). Hence the spectral sequence degenerates to yield \(H^n(N/L, \mu_*^{G/H}(\mathcal{G}^{2k})) \simeq H^n(N, \mathcal{G}^{2k})\). Moreover, from conditions (i) and (ii), we see that the sheaf \(\mu_*^{G/H}(\mathcal{G}^{2k}) \to B\) is identical with \(\mathcal{G}^{2k} \to B\). Hence we get

\[
H^n(N/L, \mathcal{G}^{2k}) \simeq H^n(N, \mathcal{G}^{2k}).
\]

For another application, see the proof of [3, Corollary 4.3].

**Bibliography**


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