PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 1, October 1979

A MULTIPLIER THEOREM FOR ANALYTIC FUNCTIONS OF SLOW MEAN GROWTH

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ABSTRACT. Multipliers into l^q spaces are characterized for certain spaces of functions analytic in the unit disk whose p-means ($0 < p \leqslant 1$) do not grow too rapidly.

1. The purpose of this paper is to extend some results on multiplier sequences to certain spaces of analytic functions first studied by Hardy and Littlewood. For an exponent p ($0 < p < \infty$) and a positive number a, let H_a^p denote the class of all functions $f(z)$ analytic in the open unit disk for which

$$M_p(f, r) = O((1 - r)^{-a}), \quad r \to 1^-,$$

where, as usual,

$$M_p(f, r) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p}.$$

The multipliers from H_a^p to l^q ($0 < p \leqslant 1$) will be described in §3. To do this, an invariant metric will be introduced on H_a^p in §2 and it will be shown that H_a^p is complete with respect to this metric. It will follow that H_a^p is an F-space, and hence the tools of functional analysis, in particular the closed graph theorem will become available.

If X is a set of functions analytic in the unit disk, then every function f in X has a Taylor series expansion $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$. A sequence $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ of complex numbers is called a multiplier of X into l^q, written $\lambda \in M(X, l^q)$ if the sequence $\{\lambda_n \hat{f}(n)\}_{n=0}^{\infty}$ belongs to l^q for every f in X. Multiplier sequences have been studied extensively by many authors. In particular Duren and Shields [1] described the multipliers from H^p to l^∞ ($0 < p \leqslant 1$) and H^p to l^q ($0 < p < 1, p \leqslant q < \infty$). In §3 these results will be extended to describe the multipliers from H_a^p to l^q.

The following theorems, essentially due to Hardy and Littlewood [2], will be needed in the sequel.

THEOREM A. *If f is analytic in the open unit disk, and* $0 < p < 1$, *then*

$$M_1(f, r) \leqslant M_\infty^{1-t}(f, r) M_p^t(f, r)$$

for all t, $0 \leqslant t \leqslant p$.

Received by the editors August 16, 1978.
AMS (MOS) subject classifications (1970). Primary 30A78.

THEOREM B. *If f is analytic in the open unit disk, $0 < p < 1$, $a > 0$, and*

$$M_p(f, r) \leqslant C(1 - r)^{-a}, \quad 0 \leqslant r < 1,$$

then

$$M_\infty(f, r) \leqslant AC(1 - r)^{-a-1/p}, \quad 0 \leqslant r < 1,$$

where A is a constant depending only on p and a.

THEOREM C. *If f, C, p and a are as in Theorem B, then*

$$M_1(f, r) \leqslant AC(1 - r)^{1-a-1/p}$$

where A is a constant depending only on p and a.

Theorem A is a simple consequence of Hölder's inequality and Jensen's convexity theorem. Theorem B follows from a nontrivial mean value property of functions $|u|^p$ where u is harmonic, and Theorem C is an immediate consequence of Theorems A and B.

2. For fixed p $(0 < p < \infty)$ and a $(a \geqslant 0)$ define

$$\|f\| = \|f\|_{p,a} = \begin{cases} \sup\limits_{0<r<1} (1 - r)^a M_p(f, r), & p \geqslant 1, \\ \sup\limits_{0<r<1} (1 - r)^{pa} M_p^p(f, r), & p < 1, \end{cases}$$

for f analytic in the open unit disk. The space H_a^p consists of those f for which $\|f\|_{p,a} < \infty$. It is clear that H_a^p is a linear space and that $\| \ \|_{p,a}$ defines an invariant metric in H_a^p, which is a norm if $p \geqslant 1$. Also H_0^p is the classical Hardy space H^p. It will be shown below that H_a^p is complete, hence an F-space (B-space if $p \geqslant 1$). The following lemma will be needed.

LEMMA 1. *Let $\psi_n(r)$ be a sequence of nonnegative functions on $[0, 1)$ satisfying:*

(a) $\psi_n(r) \leqslant B$ *for* $n = 1, 2, \ldots,$ $0 \leqslant r < 1$,

(b) $\lim_{n\to\infty} \sup_{0 \leqslant r<1} \psi_n(r)$ *exists and is finite,*

(c) $\lim_{n\to\infty} \psi_n(r) = \psi(r)$ *exists for each* r, $0 \leqslant r < 1$;

then

$$\sup_{0 \leqslant r<1} \psi(r) \leqslant \lim_{n\to\infty} \sup_{0 \leqslant r<1} \psi_n(r).$$

PROOF. Let $C = \lim_{n\to\infty} \sup_{0 \leqslant r<1} \psi_n(r)$ and $D = \sup_{0 \leqslant r<1} \psi(r)$; clearly $C \leqslant B$, $D \leqslant B$. Let $\varepsilon > 0$. There exists r_0, $0 \leqslant r_0 < 1$ such that

$$D - \varepsilon/3 < \psi(r_0) \leqslant D$$

and a positive integer n_0 such that

$$|\psi_n(r_0) - \psi(r_0)| < \varepsilon/3 \quad \text{if } n > n_0;$$

thus

$$\begin{aligned}\psi(r_0) &< \psi_n(r_0) + \varepsilon/3, \quad n > n_0,\\ &\leqslant \sup_{0\leqslant r<1} \psi_n(r) + \varepsilon/3, \quad n > n_0,\\ &\leqslant \varliminf_{n\to\infty} \sup_{0\leqslant r<1} \psi_n(r) + 2\varepsilon/3\\ &= C + 2\varepsilon/3.\end{aligned}$$

Therefore, $D - \varepsilon/3 \leqslant C + 2\varepsilon/3$. The lemma follows on letting $\varepsilon \to 0$.

Proposition 1. *The spaces* H_a^p, $0 < p < \infty$, $a > 0$, *are complete.*

Proof. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in H_a^p. Defining $f_{n,r}(z) = f_n(rz)$, it is easy to see that for each $r < 1$ the sequence $\{f_{n,r}\}_{n=1}^\infty$ is a Cauchy sequence in the Hardy space H^p. In particular the sequence $\{f_n\}_{n=1}^\infty$ converges uniformly on compact subsets of the open unit disk. Define $f(z) = \lim_{n\to\infty} f_n(z)$ for $|z| < 1$. Clearly $\|f_n\|$ converges to some finite limit, say $\lim_{n\to\infty} \|f_n\| = B$, and for each r, $0 \leqslant r < 1$, $M_p(f - f_n, r) \to 0$ as $n \to \infty$. Hence for $1 \leqslant p$, $M_p(f, r) = \lim_{n\to\infty} M_p(f_n, r)$, so

$$\begin{aligned}(1 - r)^a M_p(f, r) &= (1 - r)^a \lim_{n\to\infty} M_p(f_n, r)\\ &\leqslant \lim_{n\to\infty} \|f_n\| = B, \quad 0 \leqslant r < 1.\end{aligned}$$

Thus, $\|f\|_{p,a} \leqslant B < \infty$, so $f \in H_a^p$. If $0 < p < 1$,

$$\begin{aligned}M_p^p(f, r) &\leqslant \lim_{n\to\infty} M_p^p(f - f_n, r) + \lim_{n\to\infty} M_p^p(f_n, r)\\ &= \lim_{n\to\infty} M_p^p(f_n, r),\end{aligned}$$

and as above, $\|f\|_{p,a} \leqslant B < \infty$, so $f \in H_a^p$. It follows from Lemma 1 applied to the functions $\psi_n(r) = (1 - r)^a M_p(f_n - f_m, r)$, that for each fixed m,

$$\sup_{0\leqslant r<1} \lim_{n\to\infty} (1 - r)^a M_p(f_n - f_m, r) \leqslant \lim_{n\to\infty} \sup_{0\leqslant r<1} (1 - r)^a M_p(f_n - f_m, r).$$

Therefore

$$\begin{aligned}\|f - f_m\| &= \sup_{0\leqslant r<1} (1 - r)^{a(p)} M_p^{(p)}(f - f_m, r)\\ &= \sup_{0\leqslant r<1} \lim_{n\to\infty} (1 - r)^{a(p)} M_p^{(p)}(f_n - f_m, r)\\ &\leqslant \lim_{n\to\infty} \sup_{0\leqslant r<1} (1 - r)^{a(p)} M_p^{(p)}(f_n - f_m, r)\\ &= \lim_{n\to\infty} \|f_n - f_m\|,\end{aligned}$$

since

$$(1 - r)^a M_p(f - f_m, r) = \lim_{n\to\infty} (1 - r)^a M_p(f_n - f_m, r)$$

for each r, $0 \leqslant r < 1$. Here (p) is p or 1 according as $p < 1$ or $p \geqslant 1$. It follows that $\|f - f_m\| \to 0$ as $m \to \infty$. This completes the proof.

3. This section is devoted to describing the multipliers from H_a^p $(0 < p \leq 1)$ to l^q. The following three lemmas will be used. The first is due to Hardy and Littlewood [**2**]. Since the proofs are brief, they will be included.

LEMMA 2. *If $f \in H_a^p$ $(0 < p \leq 1, a > 0)$, then*

$$|\hat{f}(n)| \leq C\|f\|_{p,a}^{1/p} n^{a+1/p-1}, \quad n = 1, 2, \ldots,$$

where C is a constant depending only on p and a.

PROOF. By Cauchy's formula,

$$|\hat{f}(n)| \leq r^{-n} M_1(f, r) \leq Cr^{-n}\|f\|^{1/p}(1 - r)^{1-a-1/p}$$

by Theorem C, and the lemma follows on setting $r = 1 - 1/n$.

LEMMA 3. *For $\gamma > 0$, let $g_\gamma(z) = (1 - z)^{-\gamma}$. Then*
(a) $\hat{g}_\gamma(n) \sim n^{\gamma-1}/\Gamma(\gamma)$,
(b) $g_\gamma \in H_a^p$ $(0 < p < \infty, a > 0)$ *if* $1/p < \gamma \leq a + 1/p$.

PROOF. The first part is well known (apply Stirling's formula to the coefficients in the binomial series). The second part follows from the estimate

$$\int_{-\pi}^{\pi} (1 - 2r\cos\theta + r^2)^{\delta/2}\, d\theta = O\big((1 - r)^{1-\delta}\big), \quad \delta > 1.$$

If $p\gamma > 1$,

$$\begin{aligned} M_p^p(g_\gamma, r) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - 2r\cos\theta + r^2)^{-p\gamma/2}\, d\theta \\ &= O\big((1 - r)^{1-p\gamma}\big) = O\big((1 - r)^{-ap}\big] \quad \text{if } a \geq \gamma - 1/p. \end{aligned}$$

LEMMA 4. *If $\|f_n - f\|_{p,a} \to 0$, then $\hat{f}_n(k) \to \hat{f}(k)$ for all k.*

PROOF.

$$\begin{aligned} |\hat{f}_n(k) - \hat{f}(k)| &= \left| \frac{1}{2\pi i} \int_{|z|=1/2} (f_n(z) - f(z))/z^{k+1}\, dz \right| \\ &\leq 2^k M_1(f_n - f, 1/2) \leq B\|f_n - f\|_{p,a}. \end{aligned}$$

The main result describes the multipliers from H_a^p $(0 < p \leq 1)$ to l^q.

THEOREM 1. *The sequence $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ is a multiplier from H_a^p to l^q $(0 < p \leq 1, a > 0)$ if and only if the sequence $\{n^{a+1/p-1}\lambda_n\}_{n=0}^{\infty}$ belongs to l^q.*

PROOF. *Case $q = \infty$.* If $|\lambda_n| \leq Cn^{1-1/p-a}$ and $f \in H_a^p$, then by Lemma 2,

$$|\lambda_n \hat{f}(n)| \leq C'\|f\| n^{a+1/p-1} n^{1-a-1/p} = C'\|f\|$$

and λ is a multiplier. Conversely, if λ is a multiplier, λ defines a linear operator Λ: $H_a^p \to l^\infty$ given by $\Lambda(f) = \{\lambda_n \hat{f}(n)\}_{n=0}^{\infty}$. If $f_n \to f$ in H_a^p and $\{\lambda_k \hat{f}_n(k)\} \to \{s_k\}$ in l^∞, then $s_k = \lambda_k \hat{f}(k)$ for each k, since $\hat{f}_n(k) \to \hat{f}(k)$ for each k, by Lemma 4. Thus Λ has closed graph. Hence there exists a constant

C such that $|\lambda_k \hat{f}(k)| \leqslant C\|f\|$ for all k and all $f \in H_a^p$. Applying this inequality to the function g_γ with $\gamma = 1/p + a$ proves the theorem.

Case $q < \infty$. As above, if $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier, then an application of the closed graph theorem gives a constant C such that

$$\sum_{n=0}^{\infty} |\lambda_n \hat{f}(n)|^q \leqslant C\|f\|_{p,a}^q.$$

Applying this inequality to the function g_γ, $\gamma = 1/p + a$ proves the necessity. The sufficiency is immediate from Lemma 2. That completes the proof.

REMARKS. (1) The use of the closed graph theorem is standard in problems of this sort.

(2) That the condition on multipliers in Theorem 1 is simpler than the condition of Duren and Shields for the Hardy spaces H^p and for $q < \infty$, follows from the fact that $(1 - z)^{-1}$ does not belong to H^1, but belongs to H_a^1 for all $a > 0$. The more elementary proof here follows for the same reason.

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