OPERATORS COMMUTING WITH POSITIVE OPERATORS

MEHDI RADJABALIPOUR

Abstract. Necessary and sufficient conditions are obtained for an operator to commute with a positive operator.

Throughout the paper, by an operator we mean a bounded linear transformation acting in a Hilbert space $H$. The algebra of all operators in $H$ is denoted by $B(H)$.

Arveson's theorem [1] about transitive algebras states that if $\mathcal{A}$ is a strongly closed transitive algebra of operators and if $\mathcal{A}$ contains a maximal abelian selfadjoint algebra (with respect to $B(H)$), then $\mathcal{A} = B(H)$. (A transitive algebra is one whose only invariant subspaces are $\{0\}$ and $H$.). Foias [2] gives a different proof of Arveson's theorem mainly based on the following facts:

(F1) If $\mathcal{A}$ is a strongly closed proper subalgebra of $B(H)$, then $\mathcal{A}$ leaves the range of a nonzero, noninvertible positive operator $K$ invariant. In particular, if $\mathcal{A}$ contains a maximal abelian selfadjoint algebra $\mathfrak{A}$, then $K$ can be chosen such that $K \in \mathfrak{A}$ and $\mathfrak{A}K \subset K\mathfrak{A}$.

(F2) If $\mathcal{A}$ is a uniformly closed algebra and $\mathcal{A}K \subset K\mathcal{A}$ for some noninvertible positive operator $K \neq 0$, then $\mathfrak{A}$ is not transitive.

In the proof of (F2) it is shown that if $E$ is the resolution of the identity for $K$ and $T \in \mathcal{A}$, then $TE([t, \infty))H \subset E([t/a, \infty))H$ for $0 < t < \|K\|$, where $a$ is a fixed number not less than 1. (For a similar result about decomposable operators see [3].) In the present paper, given a positive operator $K$, we study conditions on an arbitrary $T$ (not necessarily in an algebra) to satisfy the above condition, and also prove a kind of converse to our result. In fact, Theorem 1 shows that if $T$ is an operator and $a > 1$, and if

$$\liminf_{n \to 0} \left(\frac{\|K^{-n}TK^n\|}{a^n}\right) < \infty,$$

(1)

then

$$TE([t, \infty))H \subset E([t/a, \infty))H, \quad 0 < t < \|K\|.$$

(2)

Note that we assume $K^{-n}TK^n$ can be extended boundedly to all of $H$. In (1) and (2), $K$ is an injective positive operator and $E$ is its resolution of the identity. Conversely, Theorems 2 and 3 show that if $T$ satisfies (2), then (1) holds but for $a$ replaced by $a^2$. As corollaries, we obtain necessary and
sufficient conditions for an operator to commute with an injective positive operator.

**Theorem 1.** Let $K$ be an injective positive operator and let $a > 1$. Then the following assertions are true.

(a) For every operator $T$, condition (1) implies condition (2).

(b) If condition (2) holds for all operators $T$ in some algebra $\mathcal{A}$, then for each $t \in (0, ||K||)$ the closure of $\mathcal{A} E([t, \infty))H$ is an invariant subspace of $\mathcal{A}$, which is nontrivial if $0 \in \sigma(K)$ and $I \in \mathcal{A}$.

(c) (Foias) If $\mathcal{A} K \subset K \mathcal{A}$ for some uniformly closed algebra $\mathcal{A}$, then for each $t \in (0, ||K||)$ the closure of $\mathcal{A} E([t, \infty))H$ is an invariant subspace of $\mathcal{A}$ and $K$, which is nontrivial if $0 \in \sigma(K)$ and $I \in \mathcal{A}$.

**Proof.** Assume without loss of generality that $||K|| = 1$ and $0 < t < 1$.

(a) Let $b < 1/a$ and let $K = K_1 \oplus K_2 \oplus K_3$ with respect to some orthogonal direct sum $H = H_1 \oplus H_2 \oplus H_3$ such that $\sigma(K_1) \subset [t, 1]$, $\sigma(K_2) \subset [tb, t]$ and $\sigma(K_3) \subset [0, tb]$. Note that some of these spaces may be trivial. If $H_3 \neq \{0\}$, let $T_3 = P_3 T | H_1$, where $P_3 : H \to H_3$ is the projection onto $H_3$. It follows that

$$||T_3|| = ||K_3 T_3 K_3^{-n}|| < \lim_{n \to 0} \inf ||K_n T_3 K_3^{-n}|| = 0,$$

Thus $||T_3|| < \lim \inf_{n \to 0} ||K^{-n} T K^n|| b^n$, and hence $TH_1 \subset H_1 \oplus H_2 \subset \mathcal{A} E([t, 1])H$. Therefore $TE([t, 1])H \subset \mathcal{A} E([tb, 1])H$ for all $b < 1/a$ and thus (2) follows.

(b) assume without loss of generality that $\mathcal{A}$ contains the identity. Then for $t \in (0, 1)$ the closure of $\mathcal{A} E([t, 1])H$ is a nonzero invariant subspace of $\mathcal{A}$ (included in $\mathcal{A} E([t/a, 1])H$).

(c) Here, again, assume without loss of generality that $\mathcal{A}$ contains the identity. Since $\mathcal{A}$ is uniformly closed and $K^{-1}TK \in \mathcal{A}$ for all $T \in \mathcal{A}$, it follows from the closed graph theorem that the map $W(T) = K^{-1}TK$ is a bounded operator in $\mathcal{A}$ and hence

$$\lim_{n \to 0} \sup \|K^{-n}TK^n\|/\|W^n\| < \infty \text{ for all } T \in \mathcal{A}.$$

For a fixed $t \in (0, 1)$ let $M = \mathcal{A} E([t, 1])H$. In view of (b), the closure of $M$ is an invariant subspace of $\mathcal{A}$. Let $x \in E([t, 1])H$ and let $T \in \mathcal{A}$. Let $K_1 = K^1 E([t, 1])H$ and $K_2 = K^2 E([ta, t])H$. Since $K^{-1}_1 x \in E([t, 1])H$, it follows that $(K_1 \oplus K_2)^{-1} x = (K_1 \oplus K_2)^{-1} T K K_1^{-1} x = K^{-1} TK K_1^{-1} x \in M$ and thus $(K_1 \oplus K_2)^{-1} M \subset M$. Hence the closure of $M$ is an invariant subspace of $(K_1 \oplus K_2)^{-1}$ and thus of $K_1 \oplus K_2$, because $K_1 \oplus K_2$ is Hermitian.

**Corollary 1.** Let $K$ be a nonscalar, injective positive operator and assume

$$\lim_{n \to 0} \inf ||K^{-n}TK^n|| < \infty \text{ for some operator } T.$$

Then $T$ has a nontrivial invariant subspace.
Theorem 2. Let $K$ be an injective positive operator with the resolution of the identity $E$. Assume (2) holds for some operator $T$ and some $a > 1$. Then

$$\|K^{-n}TK^n\| \leq \left(\frac{a^3}{(a^n - 1)}\right)\|T\| \quad (n = 1, 2, 3, \ldots).$$

Proof. Assume without loss of generality that $\|K\| = 1$. Let $b = 1/a$. Let $H_i = E([b, 1])H$ and $H_i = E([b^i, b^{i-1}])H$ for $i = 2, 3, \ldots$. Note that some $H_i$ may be trivial. Let $J = \{i: H_i \neq \{0\}\}$. For $i, j \in J$, let $K_i = K|H_i$ and $T_j = P_j|H_j$, where $P_j: H \to H_j$ is the projection onto $H_j$. By the hypotheses, $T_j = 0$ for $i > j + 2$. For $i < j + 1$ and $i, j \in J$ we have

$$K_i^{-n}T_jK_j = P_iK_i^{-n}TK_jP_j,$$

$$\|K_i^{-n}T_jK_j\| \leq b^{-n}b^{j-i}||T|| = b^{(j-i-1)}||T||,$$

for $n = 1, 2, \ldots$. Let $C = ((c_{ij}))$ be the matrix in which

$$c_{ij} = 0 \quad \text{if} \quad i \geq j + 2,$$

$$c_{ij} = ||T||b^{(j-i-1)} \quad \text{if} \quad i < j + 2.$$

Obviously

$$\|T\|^{-1}C = b^{-2n}S + b^{-n}I + \sum_{0 \leq k} b^{nk}(S^*)^{k+1},$$

where $S$ is a unilateral shift. Hence $\|C\| \leq \|T\|b^{-2n}/(1 - b^n)$. Since $K^{-n}TK^n = ((K_i^{-n}T_jK_j))_{i,j \in J}$ is majorized by the compression $((c_{ij}))_{i,j \in J}$ of $((c_{ij}))$, it follows from [3, Lemma 1] that

$$\|K^{-n}TK^n\| \leq \frac{b^{-2n}}{1 - b^n}||T|| = \frac{a^3}{a^n - 1}||T||$$

for $n = 1, 2, \ldots$.

The interesting case of $a = 1$ is treated in the following theorem and corollaries.

Theorem 3. Let $K$ be an injective positive operator with the resolution of the identity $E$ and let $T$ be an arbitrary operator. Assume $TE([t, \infty)) = E([t, \infty))TE([t, \infty))$ for all $t > 0$. Then $\|K^{-n}TK^n\| < 4||T||$ for $n = 1, 2, \ldots$.

Proof. Assume without loss of generality that $\|K\| = 1$. Let $b = 2^{-1/n}$ and let $H_i$, $K_i$, and $T_i$ be as in the proof of Theorem 2. Here $T_i = 0$ for $i > j + 1$. Following the proof of Theorem 2, we obtain

$$\|K^{-n}TK^n\| \leq (b^{-n} / (1 - b^n))||T|| = 4||T||, \quad n = 1, 2, \ldots.$$
Corollary 3. Let $K$ be an injective positive operator and let $T$ be an arbitrary operator. Then $TK = KT$ if and only if $\| K^{-n}TK^n \|$ is uniformly bounded for $n = \pm 1, \pm 2, \pm 3, \ldots$.

The last corollary has the following generalization.

Corollary 4. Let $K$ and $L$ be two injective positive operators and let $T$ be an arbitrary operator. Then $KT = TL$ if and only if $\| K^{-n}TL^n \|$ is uniformly bounded for $n = \pm 1, \pm 2, \pm 3, \ldots$.

Proof. Consider the operators

$\begin{pmatrix} K & 0 \\ 0 & L \end{pmatrix}$ and $\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$.

and apply Corollary 3.


References


Department of Mathematics, University of Northern Iran, P. O. Box 444, Babolsar, Iran