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AN EXTENSION OF THE HARDY-LITTLEWOOD INEQUALITY

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ABSTRACT. The Hardy-Littlewood inequality is extended from L^2 to L^2_w where w is any positive nondecreasing function.

In this note we establish the inequality

$$\left(\int_J |y'|^2 w\right)^2 \leqslant 4\int_J |y|^2 w \int_J |y''|^2 w \tag{1}$$

for any complex valued function y satisfying

$$y \in L^2_w(J), \quad y' \text{ locally absolutely continuous}, \quad y'' \in L^2_w(J). \tag{2}$$

Here $J = (0, \infty)$ or $J = (-\infty, \infty)$ and w is any positive nondecreasing function on J. The case $w(t) \equiv 1$ is the inequality of the title; in this case the constant 4 is best possible when $J = (0, \infty)$ but can be replaced by 1 when $J = (-\infty, \infty)$ [3].

PROOF. Suppose y satisfies (2). First we show that

$$y^2(t)w(t) \to 0 \quad \text{as } t \to \infty. \tag{3}$$

Since $w(t)$ is nondecreasing we have

$$w(t)\int_t^\infty |y^{(i)}|^2 \leqslant \int_t^\infty |y^{(i)}|^2 w, \qquad i = 0, 2. \tag{4}$$

Hence y and y'' are in $L^2(t, \infty)$ and by a well-known result [2] relating the supremum norm of y' to the L^2 norms of y and y'' we have

$$\begin{aligned}|y(t)|^2 \leqslant \|y\|^2_{\infty,(t,\infty)} &\leqslant K\left(\int_t^\infty |y|^2\right)^{1/4}\left(\int_t^\infty |y''|^2\right)^{3/4} \\ &\leqslant Kw^{-1}(t)\left(\int_t^\infty |y|^2 w\right)^{1/4}\left(\int_t^\infty |y''|^2 w\right)^{3/4}.\end{aligned}$$

Therefore $w(t)y^2(t) \to 0$ as $t \to \infty$.

Now we show that the differentiation operator A defined by $Ay = y'$ is dissipative on $L_w(0, \infty)$. We have, for $y = u + iv$,

$$\operatorname{Re}(Ay, y) = \int_0^\infty (u'u + v'v)w$$

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and

$$2\int_0^t u'uw = u^2(t)w(t) - u^2(0)w(0) - \int_0^t u^2\,dw. \tag{5}$$

From (3) and (5) we conclude that $\mathrm{Re}(Ay, y) \leqslant 0$ for all y satisfying (2), i.e., y in the domain of A^2 on $L_w^2(0, \infty)$ and hence A is dissipative. Since every dissipative operator on Hilbert space has an m-dissipative extension **[1]** inequality (1) follows from Kato's inequality for m-dissipative operators **[4]**. The proof for $J = (-\infty, \infty)$ is entirely similar.

Inequality (1) does not hold for arbitrary weight functions w as can be seen from the simple example: $y(t) = t$, $w(t) = e^{-t}$, $0 \leqslant t < \infty$.

Kato in **[4]** also characterizes the cases of equality. From this characterization follows that there is equality in (1) if and only if

$$\int_J f_{b,c} f''_{b,c} w = 0 \tag{6}$$

for some constants b, c, where

$$f_{b,c}(t) = \exp(-bt/2)\sin(\sqrt{3}\,bt/2 - c).$$

When $w = 1$ and $J = (0, \infty)$, all extremals of (1) are given by $af_{b,c}$ with $c = \pi/3$ and a, b constants with $b > 0$. It can be seen from (6) that there are nonconstant weight functions w for which (1) has extremals yielding the best constant 4 in both cases $J = (0, \infty)$ and $J = (-\infty, \infty)$. For instance $w(t) = 1$ in $[0, 4\pi/\sqrt{3}\,]$ and $w(t) = 2$ for $t > 4\pi/\sqrt{3}$ has extremal

$$f(t) = \exp(-t/2)\sin(\sqrt{3}\,t/2 - \pi/3)$$

giving the constant 4 in (1). Of course when (6) does not hold the best constant in (1) may be less than 4.

References

1. R. Beals, *Topics in operator theory*, Univ. of Chicago Press, Chicago, Ill., 1971.

2. V. N. Gabušin, *Inequalities for norms of a function and its derivatives in L^p metrics*, Mat. Zametki **1** (1967), 291–298. (Russian)

3. G. H. Hardy, J. E. Littlewood and G. Polyá, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

4. T. Kato, *On an inequality of Hardy, Littlewood and Polyá*, Advances in Math. **7** (1971), 217–218.

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