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SIMULTANEOUS IDEMPOTENTS IN $\beta N \setminus N$ AND FINITE SUMS AND PRODUCTS IN N

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ABSTRACT. The principal result is that there do not exist simultaneous multiplicative and additive idempotents in $\beta N \setminus N$. Some consequences of the (already known) existence of multiplicative idempotents which are close to additive idempotents are also derived.

1. Introduction. Around 1971, F. Galvin wanted to know if there existed ultrafilters p on N such that $\{x \in N: A - x \in p\} \in p$ whenever $A \in p$ (where $A - x = \{y \in N: x + y \in A\}$). Galvin called such ultrafilters "almost translation invariant". His motivation was that he knew that the existence of such an ultrafilter would imply the validity of a conjecture, made independently by Sanders [**10**] and by Graham and Rothschild [**6**]. This conjecture was that, whenever $r \in N$ and $N = \bigcup_{i=1}^{r} A_i$, there exist i and infinite $B \subseteq A_i$ such that $\sum F \in A_i$ whenever $F \in \operatorname{fin}(B)$ (where $\operatorname{fin}(B) = [B]^{<\omega} \setminus \{\varnothing\}$, the finite nonempty subsets of B).

The conjecture was proved in 1972 [**7**] and as a consequence Galvin's almost translation invariant ultrafilters were known to exist, provided the continuum hypothesis was assumed [**9**].

In 1975 Glazer proved directly, without assumption of the continuum hypothesis, that Galvin's almost translation invariant ultrafilters exist [**5**] (see also [**2**]). This result provided an independent proof of the above mentioned conjecture. (There are now a total of four such proofs. For the others see [**1**] and [**4**].)

Glazer's approach was to define the sum of two ultrafilters p and q on N by

$$p + q = \{A \subseteq N: \{x \in N: A - x \in p\} \in q\}.$$

(Glazer says that this definition is implicitly contained in some work of Ellis.) Then, using methods of topological dynamics, Glazer showed that there exist additive idempotents in $\beta N \setminus N$ (viewed here as the set of nonprincipal ultrafilters on N). Such an additive idempotent is exactly an almost translation invariant ultrafilter.

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Defining analogously $p \cdot q = \{A \subseteq N\colon \{x \in N\colon A/x \in p\} \in q\}$ (where $A/x = \{y \in N\colon y \cdot x \in A\}$), one obtains in a similar fashion a multiplicative idempotent. (Van Douwen [3] has obtained several interesting results about extensions to βN of any binary operation on N.) From the existence of a multiplicative idempotent in $\beta N \setminus N$ one obtains a proof of the fact that, whenever $r \in N$ and $N = \bigcup_{i=1}^{r} A_i$, there exist i and infinite $B \subseteq A_i$ such that $\prod F \in A_i$ whenever $F \in \operatorname{fin}(B)$.

For $B \subseteq N$, let $\mathrm{FS}(B) = \{\sum F\colon F \in \operatorname{fin}(B)\}$ and let $\mathrm{FP}(B) = \{\prod F\colon F \in \operatorname{fin}(B)\}$. A natural question arises: If $r \in N$ and $N = \bigcup_{i=1}^{r} A_i$, must there exist i and infinite $B \subseteq A_i$ such that $\mathrm{FS}(B) \cup \mathrm{FP}(B) \subseteq A_i$? This question is still open.[2] It is known [8, Theorem 2.13] that the existence of a simultaneous idempotent in $\beta N \setminus N$ (i.e., a nonprincipal ultrafilter p on N such that $p \cdot p = p$ and $p + p = p$) would imply an affirmative answer–in fact a much stronger result.

In [8] it was shown that there is a multiplicative idempotent p in $\beta N \setminus N$ such that

$$p \in \mathrm{cl}_{\beta N}\{q \in \beta N\colon q + q = q\}$$

and consequently that, whenever $r \in N$ and $N = \bigcup_{i=1}^{r} A_i$, there exist i and infinite subsets B and C of A_i such that $\mathrm{FS}(B) \cup \mathrm{FP}(C) \subseteq A_i$. Additional consequences of the existence of such ultrafilters will be presented here in §3.

The existence of multiplicative idempotents in $\beta N \setminus N$ which are topologically close to the set of additive idempotents made it seem likely that there might exist some simultaneous idempotent. We present here in §2 a proof that simultaneous idempotents do not exist in $\beta N \setminus N$.

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2. The nonexistence of simultaneous idempotents. The proof that there do not exist simultaneous idempotents in $\beta N \setminus N$ is completely elementary. Very few facts about ultrafilters on N are needed; we will make this presentation entirely self-contained.

The points of $\beta N \setminus N$ are the nonprincipal ultrafilters on N; p is a nonprincipal ultrafilter on N if and only if $p \subseteq \mathcal{P}(N)$, each member of p is infinite, p is closed under finite intersections, and whenever $r \in N$ and $N = \bigcup_{i=1}^{r} A_i$ one has some $A_i \in p$.

If $p \in \beta N \setminus N$ and $p + p = p$, then

$$p = \{A \subseteq N\colon \{x \in N\colon A - x \in p\} \in p\}$$

so that, whenever $A \in p$ there exists x in A such that $A - x \in p$. Similarly if $p \in \beta N \setminus N$ and $p \cdot p = p$, then whenever $A \in p$ there exists x in A such that $A/x \in p$. As a final preliminary, we note that if $p \in \beta N \setminus N$, $p + p = p$,

[2] The author has recently answered this question in the negative.

$r \in N$, and $M(r) = \{x \cdot r : x \in N\}$, then $M(r) \in p$. (To see this, note that for some i in $\{0, 1, \ldots, r-1\}$ we have $M(r) + i \in p$. But if $x \in M(r) + i$, then $(M(r) + i) - x = M(r)$.)

2.1 LEMMA. *Let $p \in \beta N \setminus N$ such that $p + p = p$. If there is a family $\{A_n : n \in N$ and $n \geqslant 2\} \subseteq p$ such that for all $n \geqslant 2$ and all $m \geqslant 2$ one has $(n \cdot A_n) \cap (m + A_m) = \varnothing$, then $p \cdot p \neq p$.*

PROOF Let $B = \bigcup_{m=2}^{\infty}(m + A_m)$. We first claim that $B \in p$. Suppose instead that $N \setminus B \in p$ and pick $m \geqslant 2$ in $N \setminus B$ such that $(N \setminus B) - m \in p$. Pick x in $((N \setminus B) - m) \cap A_m$. Then $x + m \in N \setminus B$ while $x + m \in m + A_m$, a contradiction.

Suppose now that $p \cdot p = p$, and pick $n \geqslant 2$ in B such that $B/n \in p$. Pick x in $(B/n) \cap A_n$. Then $x \cdot n \in B$, so pick $m \geqslant 2$ such that $x \cdot n \in m + A_m$. Then $x \cdot n \in (m + A_m) \cap (n \cdot A_n)$, a contradiction.

We omit the easy verification of the following arithmetic lemma.

2.2 LEMMA. *For r and k in N, let $a_{0,k,r} = k^{2r-1} + k^{r-1}$, $a_{1,k,r} = b_{0,k,r} = k^{2r} + k^{r-1}$, $a_{2,k,r} = b_{1,k,r} = k^{2r} + k^r$, and $b_{2,k,r} = a_{0,k,r+1}$. If $k \geqslant 2$, $r \geqslant 4$, and $i \in \{0, 1, 2\}$, then*

$$k \cdot b_{i,k,r} \leqslant a_{i,k,r+1} \quad \text{and} \quad (k \cdot a_{i,k,r} - b_{i,k,r})^3 > b_{i,k,r}.$$

2.3 THEOREM. *Let $p \in \beta N \setminus N$. Then $p + p \neq p$ or $p \cdot p \neq p$.*

PROOF. We assume that $p + p = p$ and apply Lemma 2.1. For each i in $\{0, 1, 2\}$ and each k in N with $k \geqslant 2$, let

$$B_{i,k} = \bigcup_{r=4}^{\infty} \{x \in N : a_{i,k,r} \leqslant x < b_{i,k,r}\}.$$

Given k in N with $k \geqslant 2$, we have

$$N = \bigcup_{i=0}^{2} B_{i,k} \cup \{x \in N : x < a_{0,k,4}\}.$$

Since the latter set is finite, we have $f(k)$ in $\{0, 1, 2\}$ such that $B_{f(k),k} \in p$. For $n \geqslant 2$, let

$$A_n = \bigcap_{k=2}^{n} \left(B_{f(k),k} \cap M(k^3) \cap M(k+1)\right).$$

Then $\{A_n : n \in N$ and $n \geqslant 2\} \subseteq p$.

We claim that for $n \geqslant 2$ and $m \geqslant 2$, $(n \cdot A_n) \cap (m + A_m) = \varnothing$. Suppose not, and pick $n \geqslant 2$, $m \geqslant 2$, x in A_n, and y in A_m such that $n \cdot x = m + y$. We consider first the possibility that $m < n$. Then $A_n \subseteq A_m$ so we have $x \in M(m+1)$ and $y \in M(m+1)$. Consequently $m \in M(m+1)$, a contradiction.

Thus we have $m \geqslant n$. Let $i = f(n)$ and note that for some $r \geqslant 4$ and some $s \geqslant 4$ we have $a_{i,n,r} \leqslant x < b_{i,n,r}$ and $a_{i,n,s} \leqslant y < b_{i,n,s}$ since both x and y are in

$B_{i,n}$. Since $x < b_{i,n,r}$ we have, by Lemma 2.2, that $y < n \cdot x < a_{i,n,r+1}$ and hence $s \leqslant r$. Thus $y < b_{i,n,r}$. Also $x \geqslant a_{i,n,r}$ so $m = nx - y > n \cdot a_{i,n,r} - b_{i,n,r}$. So by Lemma 2.2, $m^3 > b_{i,n,r}$. But $y \in M(m^3)$ so $b_{i,n,r} < y$, a contradiction.

3. Sums and products within cells of a partition of N. As earlier remarked, it was shown **[8]** that there is a multiplicative idempotent p in $\beta N \setminus N$ such that $p \in \mathrm{cl}_{\beta N}\{q \in \beta N\colon q + q = q\}$ and consequently that, whenever $r \in N$ and $N = \bigcup_{i=1}^{r} A_i$ one has some i and some B and C in $[A_i]^\omega$ such that $\mathrm{FS}(B) \cup \mathrm{FP}(C) \subseteq A_i$. ($[A]^k = \{B \subseteq A\colon |B| = k\}$.) (These conclusions are drawn from Corollary 2.11 and the proof of Theorem 2.6 in **[8]**.) We present here some stronger results which follow from the existence of such an ultrafilter and some discussion of the finite versions of the main open question.

3.1 THEOREM. *Let $p \in \beta N \setminus N$ such that $p \cdot p = p$ and $p \in \mathrm{cl}_{\beta N}\{q \in \beta N\colon q + q = q\}$ and let $A \in p$.*

(a) *There is an increasing sequence $\langle x_n \rangle_{n=1}^{\infty}$ in A such that, whenever $F \in \mathrm{fin}(N)$ and $m = \min F$ one has some C in $[A]^m$ such that $\prod_{n \in F} x_n = \sum C$ and $\mathrm{FS}(C) \subseteq A$;*

(b) *for each m in N there exists B in $[A]^\omega$ such that for each F in $\mathrm{fin}(B)$ one has some C in $[A]^m$ with $\sum F = \prod C$ and $\mathrm{FP}(C) \subseteq A$;*

(c) *there exists B in $[A]^\omega$ such that $\mathrm{FS}(B) \subseteq A$ and, for each x in $\mathrm{FS}(B)$, one has some C in $[A]^\omega$ such that $x = \min C$ and $\mathrm{FP}(C) \subseteq A$; and*

(d) *there exists B in $[A]^\omega$ such that $\mathrm{FP}(B) \subseteq A$ and, for each x in $\mathrm{FP}(B)$, one has some C in $[A]^\omega$ such that $x = \min C$ and $\mathrm{FS}(C) \subseteq A$.*

PROOF. As in the proof of Theorem 2.6 of **[8]** we have that, whenever $D \in p$ one has some B and C in $[D]^\omega$ such that $\mathrm{FS}(B) \cup \mathrm{FP}(C) \subseteq D$.

(a) For each n in N, let $E_n = \{x \in N\colon$ there exists C in $[A]^n$ such that $\mathrm{FS}(C) \subseteq A$ and $x = \sum C\}$. We claim that each $E_n \in p$. Suppose instead that $A \setminus E_n \in p$ and choose D in $[A \setminus E_n]^\omega$ such that $\mathrm{FS}(D) \subseteq A \setminus E_n$. Pick C in $[D]^n$ and let $x = \sum C$. Since $C \subseteq A$, we have $x \in E_n$. On the other hand $x \in \mathrm{FS}(C)$ so $x \notin E_n$, a contradiction.

Now let $A_1 = E_1 \cap A$ and choose x_1 in A_1 such that $A_1/x_1 \in p$. Inductively let $A_{n+1} = E_{n+1} \cap A_n \cap (A_n/x_n)$ and choose x_{n+1} in A_{n+1} such that $A_{n+1}/x_{n+1} \in p$ and $x_{n+1} > x_n$. One easily shows by induction on $|F|$ that, if $F \in \mathrm{fin}(N)$ and $m = \min F$, then $\prod_{n \in F} x_n \in A_m$.

(b) Let $m \in n$ and let $E = \{x \in N;$ there exists C in $[A]^m$ such that $\mathrm{FP}(C) \subseteq A$ and $x = \prod C\}$. As above, one has $E \in p$. Choose B in $[A \cap E]^\omega$ such that $\mathrm{FS}(B) \subseteq A \cap E$.

(c) Let $E = \{x\colon$ there exists C in $[A]^\omega$ such that $x = \min C$ and $\mathrm{FP}(C) \subseteq A\}$. As above, $E \in p$. Choose B in $[A \cap E]^\omega$ such that $\mathrm{FS}(B) \subseteq A \cap E$.

The proof of (d) is similar.

3.2 COROLLARY. *If $\mathcal{Q}$ is a finite partition of N, then there exists A in $\mathcal{Q}$ such that each of the conclusions of Theorem* 3.1 *holds.*

By comparison with the strong infinite results presented above, it is rather shocking to note the lack of knowledge about the simplest finite versions. (See §4 of [**8**].) In particular it is not known if, whenever $r \in N$ and $N = \bigcup_{i=1}^{r} A_i$, there exist i and arbitrarily large distinct x and y such that $\{x, y, x + y, x \cdot y\} \subseteq A_i$. (In fact if $r = 2$ one can read "arbitrarily large" as "at least 3" and if $r \geqslant 3$ one can dispense with "arbitrarily large" entirely.)

As a corollary of 3.1 (c) or (d) one can always obtain, when $N = \bigcup_{i=1}^{r} A_i$ some i and arbitrarily large distinct x, y, and z such that $\{x, y, z, x + y, x \cdot z\} \subseteq A_i$. We are grateful to R. Graham for permission to present the only other result in this direction which we know of.

3.3 THEOREM (GRAHAM). *Let $N = A_1 \cup A_2$. There exist i in $\{1, 2\}$ and, for each k in N, distinct x and y larger than k such that $\{x + y, x \cdot y\} \subseteq A_i$.*

PROOF. We show that for each k in N we can find i in $\{1, 2\}$ and distinct x and y larger than k such that $\{x + y, x \cdot y\} \subseteq A_i$. The result follows from the pigeon hole principle.

Let k be given. If, for some d we have i in $\{1, 2\}$ such that $\{3x: x \geqslant d\} \subseteq A_i$, the result is trivial. Otherwise pick $d > k$ such that $3d + 3$ and $3d + 6$ are in different cells. Pick i such that $2d(d + 3) \in A_i$. If $3d + 3 \in A_i$, let $x = d + 3$ and let $y = 2d$. If $3d + 6 \in A_i$ let $x = d$ and let $y = 2d + 6$.

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