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REGULAR SEQUENCES AND POWERS OF IDEALS

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ABSTRACT. Let I be an ideal of a Noetherian local ring. In this note we discuss how a power of I reflects that I is generated by a regular sequence.

Introduction. One way to analyze whether an ideal I of the local ring R is a complete intersection–that is, generated by a regular sequence of elements–is via the associated graded ring $\mathrm{gr}_I R = \oplus_i I^i/I^{i+1}$. A theorem of Rees then says that I is a complete intersection iff $\mathrm{gr}_I R$ is a polynomial ring over R/I. Here we will discuss cases where the behaviour of a single component I^s/I^{s+1}, $s > 1$, suffices for a converse.

THEOREM. *The following conditions are equivalent for an ideal I of a local ring R.*

(a) *I is a complete intersection.*

(b) *I has finite projective dimension and I^s/I^{s+1} is R/I-free for some $s > 0$.*

(c) *I is generically a complete intersection, grade I = height I, and I^s/I^{s+1} is R/I-free for some $s > 0$.*

The expression 'generically a complete intersection' means that if U denotes the multiplicative set of elements that are regular modulo I, then I_U is generated by a regular sequence of elements of R_U. Otherwise our terminology will be standard as in [3].

We are grateful to Judy Sally for various enlightenments.

The proof. The proof is a mix of some 'old' homological algebra and some recent work of Sally [5] and Eakin-Sathaye [1] on the combinatorics of the powers of an ideal.

There is no harm in assuming throughout that the residue field R/M of the local ring R is infinite.

Suppose $I/I^2 = (R/I)^t \oplus T$ with t = maximum rank of a direct summand of I/I^2; denote $G = (R/I)^t$. There is a natural surjection of the sth symmetric power of I/I^2 onto I^s/I^{s+1}:

$$S^s(I/I^2) = \oplus \sum S^i(G) \otimes S^{s-i}(T) \to I^s/I^{s+1} \to 0.$$

A term such as $S^i(G) \otimes S^{s-i}(T)$, $i \neq s$, cannot be a generator for the category of R/I-modules since $S^j(T)$, $j \neq 0$, is a generator iff $\otimes^j T$ is a

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generator and this module is, in turn, a quotient of a direct sum of copies of T.

Assume that I^s/I^{s+1} is R/I-free. Now, after taking exterior powers of the appropriate rank or, completing and using the Krull-Schmidt theorem, it follows that

$$\operatorname{rank}(I^s/I^{s+1}) \leq \operatorname{rank} S^s(G) = \binom{s+t-1}{t-1}.$$

On the other hand, as $\binom{s+t-1}{t-1} < \binom{s+t}{t}$, by the reduction theorem of [1]:

$$I^s = (x_1, \ldots, x_t)I^{s-1} \qquad (*)$$

for $x_i \in I$. In particular, height $I \leq t$.

(b) $\Rightarrow$ (a). By [6] I contains a regular sequence of length t and, if $T \neq 0$, a regular sequence of length $t + 1$. Thus I is generated by a regular sequence.

(c) $\Rightarrow$ (a). If height $I = r$, we may assume the existence of a regular sequence $x_1, \ldots, x_r$ in I such that $\{x_1, \ldots, x_r\}$ is part of a minimal reduction for I and generically generate I [4], [5]. Note that the monomials in the x_i's of a given degree d are linearly independent modulo MI^d [loc. cit.]. In the current situation, since $\operatorname{rank}(I^s/I^{s+1}) = \binom{r+s-1}{r-1}$ it follows that $I^s = J^s$, with $J = (x_1, \ldots, x_r)$.

We claim that $I^{s+1} = J^{s+1}$. Indeed, if $I = (J, L)$, then $I^{s+1} = (J^{s+1}, J^sL) = J(J^s, J^{s-1}L) = J^{s+1}$. We have then the isomorphism: $I^s/I^{s+1} \cong J^s/J^{s+1}$. Since the x_i's form a regular sequence, J^s/J^{s+1} is a free R/J-module, and thus J is also the annihilator of I^s/I^{s+1}.

REMARKS. (1) The condition that height I = grade I cannot always be deleted in (c). For instance, if $R = [[x, y, z]]$, $x^2 = xy = xz = 0$, $I = (x, y)$, then $I^s/I^{s+1} \cong R/I$ for $s > 1$ and I is also generically a complete intersection. On the other hand, for $s = 1$, as shown in [2, Corollary 4], the grade condition is not needed.

(2) Despite counterexamples as above, the estimate provided by (*) often suffices to conclude: I^s/I^{s+1} R/I-free $\Rightarrow I/I^2$ R/I-free. Here are two examples: (i) height $I = \nu(I)$ = minimum number of generators of I. In this case (*) shows that I^s/I^{s+1} must be free of rank $\binom{t+s-1}{t-1}$. (ii) $\nu(I) \leq 1 +$ grade I. If $T \neq 0$, t = grade I, and as in (b) $\Rightarrow$ (a) we conclude that I is generated by a regular sequence.

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