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ISOMETRIES ON L^p SPACES AND COPIES OF l^p SHIFTS

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ABSTRACT. Necessary and sufficient conditions are given on an isometry V in an L^p space so that there exists an invariant subspace M such that V restricted to M is isometrically equivalent to the unilateral shift on l^p.

1. Introduction. The unilateral shift on l^p is probably the best known and studied of the nonunitary isometries on L^p spaces. Even on l^p, isometries V such that $\bigcap V^n l^p = \{0\}$ need not be isometric to the unilateral shift on l^p. This note will characterize when an isometry on L^p contains (in the appropriate sense) a copy of the unilateral shift on l^p. The question of in what sense isometries in more general Banach spaces look like shifts is discussed in [1].

In what follows L^p will be $L^p(X, \Sigma, \mu)$ where (X, Σ, μ) is a σ-finite measure space, $1 < p < \infty, p \neq 2$. l^p is, of course, L^p with X the nonnegative integers, Σ all subsets of X, and μ counting measure. The unilateral shift S on l^p is given by $S(\alpha_0, \alpha_1, \ldots) = (0, \alpha_0, \alpha_1, \ldots)$. V will always denote an isometry on L^p. From [2] there exists a regular set isomorphism $T: \Sigma \to \Sigma$ so that

$$(Vf)(x) = h(x)(T(f))(x). \tag{1}$$

Formula (1) should be interpreted in the following sense.

If $C \in \Sigma$, let 1_C denote its characteristic function. Then $T(1_C) = 1_{T(C)}$ and T is extended to simple functions by linearity. The map V as expressed in (1) is an isometry on L^p simple functions and hence extends to all of L^p. Without loss of generality, we may assume $\mu(\{x|h(x) = 0\}) = 0$. Note that $T(1_C)$ need not be in L^p even if 1_C is. The only requirement is that $\|h1_{T(C)}\| = (\mu(C))^{1/p}$, so that $hT(1_C)$ is in L^p. It should also be pointed out that the characterization of h in [2] is vague. It is not necessary that $|h| = d\mu^*/d\mu$, $\mu^* = \mu \circ T^{-1}$. It is sufficient that

$$\int_{T(A)} |h|^p d\mu = \int_{T(A)} \frac{d\mu^*}{d\mu} d\mu, \qquad A \in \Sigma.$$

If $T(\Sigma)$ is a properly smaller Boolean algebra, then one need not have $|h|^p = d\mu^*/d\mu$.

All statements about sets and functions are modulo sets of μ-measure zero. For $f \in L^p$, supp(f) denotes its support.

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2. Main results. The main result of this note is the following characterization of when an isometry on L^p 'contains' a copy of S on l^p.

THEOREM 1. *Suppose that V: $L^p \to L^p$ is an isometry. Then the following statements* (I)–(III) *are equivalent.*

(I) *There exists a subspace $M \subseteq L^p$ such that*:

(i) $VM \subseteq M$,

(ii) *M is isometric to l^p*,

(iii) *$US = VU$ where U is the isometric map of* (ii), U: $l^p \to M$;

(II) *There exists a set $A \in \Sigma$ such that $T(A) \subset A$, $0 < \mu(A \setminus T(A)) < \infty$*;

(III) *There exists an $f \in L^p$ such that* $\operatorname{supp}(Vf) \subseteq \operatorname{supp}(f)$ *and* $0 < \mu(\operatorname{supp}(f) \setminus \operatorname{supp}(Vf)) < \infty$.

PROOF. We shall prove (I) $\Rightarrow$ (II) first. Assume that (I) holds. Let $e_i = 1_{\{i\}} \in l^p$. Let $A_i = \operatorname{supp}(U1_{\{i\}})$. Note that $A_i \cap A_j = \varnothing$ if $i \neq j$ since

$$\|Ue_i \pm Ue_j\|^p = \|e_i \pm e_j\|^p = \|e_i\|^p + \|e_j\|^p = \|Ue_i\|^p + \|Ue_j\|^p,$$

implies $Ue_i Ue_j = 0$ almost everywhere μ [2]. Note also that from (iii) $T(A_i) \subseteq A_{i+1}$. If $\mu(A_1) < \infty$, let $A = \bigcup A_i$ and (II) holds. If A_1 is not of finite measure, let E_1 be a subset of A_1 such that $\mu(E_1) < \infty$. (E_1 exists since A_1 is the support of an L^p function.) Let $A = E_1 \cup \sum_{n=1}^{\infty} T^n(E_1)$. Then (II) holds. To see that (II) $\Rightarrow$ (I), assume that (II) holds. First we need to show that:

$$\text{If } f \in L^p \text{ and } \operatorname{supp}(f) \subseteq D, \text{ then } \operatorname{supp}(V^n f) \subseteq T^n(D) \text{ for } n \geq 1. \quad (2)$$

To prove (2), it suffices to prove the $n = 1$ case. Then $n > 1$ follows by induction. If $g \in L^p$ is a simple function, so that

$$g = \sum_1^n a_i 1_{E_i}, \qquad E_i \cap E_j = \varnothing \quad \text{if } i \neq j, \quad a_i \neq 0,$$

then

$$Vg = \sum_1^n a_i V 1_{E_i} = \sum_1^n a_i h(x) 1_{T(E_i)}.$$

(Note that $T(E_i) \cap T(E_j) = \varnothing$.) Thus $\operatorname{supp}(Vg) = \bigcup T(E_i) \subseteq T(D)$ if $\bigcup_i E_i \subseteq D$. Thus (2) holds for simple functions. Now if $f \in L^p$, chose simple functions f_n so that $f_n \to f$ in L^p norm and $\operatorname{supp}(f_n) \subseteq \operatorname{supp}(f) \subseteq D$. But $Vf_n \to Vf$. Take a subsequence so that $Vf_{n_j} \to Vf$ almost everywhere. But $\operatorname{supp}(Vf_{n_j}) \subseteq T(D)$. Thus $\operatorname{supp}(Vf) \subseteq T(D)$ and (2) follows. By a similar argument using simple functions it is straightforward to show:

$$\text{If } f, g \in L^p \text{ and } \operatorname{supp}(g) = \operatorname{supp}(f), \text{ then } \operatorname{supp}(Vg) = \operatorname{supp}(Vf). \quad (3)$$

Now to prove (I). By assumption there is an $A \in \Sigma$ such that $T(A) \subseteq A$, $0 < \mu(A \setminus T(A)) < \infty$. Let $B = A \setminus T(A)$. Since $T(A) \subseteq A$, $T^n(A) \subseteq T(A)$ for $n \geq 1$. Thus $B \cap T^n(B) = \varnothing$. Since T is regular, this implies that $T^k(B) \cap T^l(B) = \varnothing$ for $k \neq l$. Let $g = \alpha h 1_B$ where $\alpha = \|h 1_B\|^{-1}$. From (2), (3), it follows that $\operatorname{supp}(V^n g) \cap \operatorname{supp}(V^n f) = \varnothing$ for $m \neq n$. Let $g_n = V^n g$. For $\sum \alpha_n e_n \in l^p$, define U: $l^p \to L^p$ by $U(\sum \alpha_n e_n) = \sum a_n g_n$. Let $M = Ul^p$. Since $\operatorname{supp}(g_n) \cap \operatorname{supp}(g_m) = \varnothing$ if $n \neq m$ and $\|g_n\| = 1$, U is an isometry so that

(ii) holds. That (i) and (ii) hold is clear and (I) follows. That (III) is equivalent to (I) and (II) is now clear. □

3. Comments. A natural question is whether (I) always holds; that is, whether every isometry on an L^p space contains a copy of the unilateral shift on l^p. If V is unitary, the answer is, in general, no. The identity is an example. What about nonunitary isometries? Do they always satisfy (I)? We shall show that the answer is no for general L^p and affirmative for nonunitary isometries on l^p.

EXAMPLE. Let $X = [0, 2]$, and μ be Lebesgue measure. For $A \in \Sigma$, define $T(A) = \frac{1}{2}A \cup \frac{1}{2}A + 1$. (Here $\frac{1}{2}A = \{\frac{1}{2}a\colon a \in A\}$, $\frac{1}{2}A + 1 = \{\frac{1}{2}a + 1\colon a \in A\}$.) Clearly T is a measure-preserving transformation of Σ into itself. Let h be identically 1 and define V by (1). Then V is an isometry of L^p into itself. Let $g = 1_{[0,1)} - 1_{[1,2)}$. Now $\int 1_{T(A)} g d\mu = 0$ for any set $A \in \Sigma$. Hence $\int (Vf) g d\mu = 0$ for all $f \in L^p$. Since $g \in (L^p)^*$, we have V is not onto and hence V is not unitary. But V cannot satisfy (II). For suppose there existed $A \in \Sigma$ such that $T(A) \subset A$, $0 < \mu(A \setminus T(A)) < \infty$. Let $B = A \setminus T(A)$. Then $2 > \mu(\bigcup T^n(B)) = \sum \mu(T^n(B)) = \sum \mu(B)$ which is impossible. Note that this V also satisfies $\bigcap R(V^n) = \{0\}$.

This example is a special case of the more general fact that:

PROPOSITION. *If $V\colon L^p \to L^p$ is an isometry and satisfies* (II), *then either $\mu(X)$ is not finite or T is not measure-preserving.*

We conclude by showing that:

THEOREM 2. *If V is a nonunitary isometry on l^p, then it satisfies* (I).

PROOF. Represent V as in (1), and assume that V is not unitary. Without loss of generality, we may assume $i \neq T^m(i)$ for all i and $m > 0$. (These unitary summands may be discarded, if present.) Pick an i. If $i \notin T^m(i)$ for all $m > 0$, then (II) holds. Thus we may assume that $i \in T^m(i)$ for some m. Take m to be the least such m. If $m = 1$, let $B = T(i) \setminus \{i\}$. Then $i \notin T(B)$ and by induction $i \notin T^m(B)$ for all m. Hence $B \cap T^m(B) = \varnothing$ for all $m > 0$ and (II) holds for $A = \bigcup T^m(C)$, C any finite subset of B. Thus we may assume $m > 1$. Now there exist distinct $i_1, \ldots, i_{m-1}$ such that $i \in T(i_{k-1})$, $i_0 = i$, $i \in T(i_{m-1})$. The distinctness of the i_j follows from the fact that m is minimal. Let $I = \{i, i_1, \ldots, i_{m-1}\}$, and $B = T(I) \setminus I$. Note that by assumption $B \neq \varnothing$. Now $I \cap C = \varnothing$ implies $I \cap T(C) = \varnothing$ for any set C. But $B \cap I = \varnothing$. Hence $T^n(B) \cap I = \varnothing$ for all $n > 0$. But then $B \cap T^n(B) = \varnothing$ for all $n > 0$ and again (II) holds. □

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