ISOMETRIES ON $L^p$ SPACES AND COPIES OF $L^p$ SHIFTS

STEPHEN L. CAMPBELL, GARY D. FAULKNER AND MARIANNE L. GARDNER

Abstract. Necessary and sufficient conditions are given on an isometry $V$ in an $L^p$ space so that there exists an invariant subspace $M$ such that $V$ restricted to $M$ is isometrically equivalent to the unilateral shift on $L^p$.

1. Introduction. The unilateral shift on $L^p$ is probably the best known and studied of the nonunitary isometries on $L^p$ spaces. Even on $L^p$, isometries $V$ such that $\bigcap V^n = \{0\}$ need not be isometric to the unilateral shift on $L^p$. This note will characterize when an isometry on $L^p$ contains (in the appropriate sense) a copy of the unilateral shift on $L^p$. The question of in what sense isometries in more general Banach spaces look like shifts is discussed in [1].

In what follows $L^p$ will be $L^p(X, \Sigma, \mu)$ where $(X, \Sigma, \mu)$ is a $\sigma$-finite measure space, $1 \leq p < \infty$, $p \neq 2$. $L^p$ is, of course, $L^p$ with $X$ the nonnegative integers, $\Sigma$ all subsets of $X$, and $\mu$ counting measure. The unilateral shift $S$ on $L^p$ is given by $S(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$. $V$ will always denote an isometry on $L^p$. From [2] there exists a regular set isomorphism $T: \Sigma \rightarrow \Sigma$ so that

$$\text{(Vf)}(x) = h(x)(T(f))(x).$$

Formula (1) should be interpreted in the following sense.

If $C \in \Sigma$, let $1_C$ denote its characteristic function. Then $T(1_C) = 1_{T(C)}$ and $T$ is extended to simple functions by linearity. The map $V$ as expressed in (1) is an isometry on $L^p$ simple functions and hence extends to all of $L^p$. Without loss of generality, we may assume $\mu(\{x|h(x) = 0\}) = 0$. Note that $T(1_C)$ need not be in $L^p$ even if $1_C$ is. The only requirement is that $\|h1_{T(C)}\| = (\mu(C))^{1/p}$, so that $hT(1_C)$ is in $L^p$. It should also be pointed out that the characterization of $h$ in [2] is vague. It is not necessary that $|h| = d\mu^*/d\mu$, $\mu^* = \mu \circ T^{-1}$. It is sufficient that

$$\int_{T(A)} |h|^p d\mu = \int_{T(A)} \frac{d\mu^*}{d\mu} d\mu, \quad A \in \Sigma.$$
2. Main results. The main result of this note is the following characterization of when an isometry on $L^p$ 'contains' a copy of $S$ on $l^p$.

**THEOREM 1.** Suppose that $V: L^p \to L^p$ is an isometry. Then the following statements (I)-(III) are equivalent.

(I) There exists a subspace $M \subseteq L^p$ such that:

(i) $VM \subseteq M$,

(ii) $M$ is isometric to $l^p$,

(iii) $US = VU$ where $U$ is the isometric map of (ii), $U: l^p \to M$;

(II) There exists a set $A \in \Sigma$ such that $T(A) \subseteq A$, $0 < \mu(A \setminus T(A)) < \infty$;

(III) There exists an $f \in L^p$ such that $\text{supp}(Vf) \subseteq \text{supp}(f)$ and $0 < \mu(\text{supp}(f) \setminus \text{supp}(Vf)) < \infty$.

**Proof.** We shall prove (I) $\Rightarrow$ (II) first. Assume that (I) holds. Let $e_i = 1_{(i)} \in l^p$. Let $A_i = \text{supp}(U1_{(i)})$. Note that $A_i \cap A_j = \emptyset$ if $i \neq j$ since

$$\|ue_i \pm ue_j\| = \|e_i \pm e_j\| = \|e_i\|^p = \|ue_i\|^p + \|ue_j\|^p,$$

implies $ue_i \pm ue_j = 0$ almost everywhere $\mu [2]$. Note also that from (iii) $T(A_i) \subseteq A_{i+1}$. If $\mu(A_i) < \infty$, let $A = \bigcup A_i$ and (II) holds. If $A_1$ is not of finite measure, let $E_1$ be a subset of $A_1$ such that $\mu(E_1) < \infty$. ($E_1$ exists since $A_1$ is the support of an $L^p$ function.) Let $A = E_1 \cup \bigcup_{n=1}^{\infty} T^n(E_1)$. Then (II) holds. To see that (II) $\Rightarrow$ (I), assume that (II) holds. First we need to show that:

If $f \in L^p$ and $\text{supp}(f) \subseteq D$, then $\text{supp}(V^n f) \subseteq T^n(D)$ for $n > 1$. (2)

To prove (2), it suffices to prove the $n = 1$ case. Then $n > 1$ follows by induction. If $g \in L^p$ is a simple function, so that

$$g = \sum_{i=1}^{n} a_i 1_{E_i}, \quad E_i \cap E_j = \emptyset \quad \text{if} \quad i \neq j, \quad a_i \neq 0,$$

then

$$Vg = \sum_{i=1}^{n} a_i 1_{T(E_i)}.$$ (Note that $T(E_i) \cap T(E_j) = \emptyset$.) Thus $\text{supp}(Vg) = \bigcup T(E_i) \subseteq T(D)$ if $\bigcup_i E_i \subseteq D$. Thus (2) holds for simple functions. Now if $f \in L^p$, chose simple functions $f_n$ so that $f_n \to f$ in $L^p$ norm and $\text{supp}(f_n) \subseteq \text{supp}(f) \subseteq D$. But $Vf_n \to Vf$. Take a subsequence so that $Vf_n \to Vf$ almost everywhere. But $\text{supp}(V^n f_n) \subseteq T(D)$. Thus $\text{supp}(Vf) \subseteq T(D)$ and (2) follows. By a similar argument using simple functions it is straightforward to show:

If $f, g \in L^p$ and $\text{supp}(g) = \text{supp}(f)$, then $\text{supp}(Vg) = \text{supp}(Vf)$. (3)

Now to prove (I). By assumption there is an $A \in \Sigma$ such that $T(A) \subseteq A$, $0 < \mu(A \setminus T(A)) < \infty$. Let $B = A \setminus T(A)$. Since $T(A) \subseteq A$, $T^n(A) \subseteq T(A)$ for $n \geq 1$. Thus $B \cap T^n(B) = \emptyset$. Since $T$ is regular, this implies that $T^n(B) \cap T^k(B) = \emptyset$ for $k \neq l$. Let $g = ah_1_B$ where $\alpha = \|h_1_B\|^{-1}$. From (2), (3), it follows that $\text{supp}(V^n g) \cap \text{supp}(V^n f) = \emptyset$ for $m \neq n$. Let $g_n = V^n g$. For $\Sigma \alpha_n e_n \in l^p$, define $U: l^p \to L^p$ by $U(\Sigma \alpha_n e_n) = \Sigma \alpha_n g_n$. Let $M = UI^p$. Since $\text{supp}(g_n) \cap \text{supp}(g_m) = \emptyset$ if $n \neq m$ and $\|g_n\| = 1$, $U$ is an isometry so that
(ii) holds. That (i) and (ii) hold is clear and (I) follows. That (III) is equivalent
to (I) and (II) is now clear. □

3. Comments. A natural question is whether (I) always holds; that is,
whether every isometry on an \( L^p \) space contains a copy of the unilateral shift
on \( l^p \). If \( V \) is unitary, the answer is, in general, no. The identity is an example.
What about nonunitary isometries? Do they always satisfy (I)? We shall show
that the answer is no for general \( L^p \) and affirmative for nonunitary isometries
on \( l^p \).

**Example.** Let \( X = [0, 2] \), and \( \mu \) be Lebesgue measure. For \( A \in \Sigma \), define
\( T(A) = \frac{1}{2} A \cup \frac{1}{2} A + 1 \). (Here \( \frac{1}{2} A = \{\frac{1}{2} a : a \in A\}, \ \frac{1}{2} A + 1 = \{\frac{1}{2} a + 1 : a \in A\} \).) Clearly \( T \) is a measure-preserving transformation of \( \Sigma \) into itself. Let \( h \)
be identically 1 and define \( V \) by (I). Then \( V \) is an isometry of \( L^p \) into itself.
Let \( g = 1_{[0,1)} - 1_{[1,2]} \). Now \( \int_{T(A)} gd\mu = 0 \) for any set \( A \in \Sigma \). Hence
\( \int (Vf)gd\mu = 0 \) for all \( f \in L^p \). Since \( g \in (L^p)^* \), we have \( V \) is not onto and
hence \( V \) is not unitary. But \( V \) cannot satisfy (II). For suppose there existed
\( A \in \Sigma \) such that \( T(A) \subset A \), \( 0 < \mu(A \setminus T(A)) < \infty \). Let \( B = A \setminus T(A) \). Then
\( 2 > \mu(\bigcup T^n(B)) = \sum \mu(T^n(B)) = \sum \mu(B) \) which is impossible. Note that this
\( V \) also satisfies \( \text{DR}(V^n) = \{0\} \).

This example is a special case of the more general fact that:

**Proposition.** If \( V : L^p \to L^p \) is an isometry and satisfies (II), then either
\( \mu(X) \) is not finite or \( T \) is not measure-preserving.

We conclude by showing that:

**Theorem 2.** If \( V \) is a nonunitary isometry on \( l^p \), then it satisfies (I).

**Proof.** Represent \( V \) as in (I), and assume that \( V \) is not unitary. Without
loss of generality, we may assume \( i \neq T^m(i) \) for all \( i \) and \( m > 0 \). (These
unitary summands may be discarded, if present.) Pick an \( i \). If \( i \not\in T^m(i) \) for
all \( m > 0 \), then (II) holds. Thus we may assume that \( i \in T^m(i) \) for some \( m \).
Take \( m \) to be the least such \( m \). If \( m = 1 \), let \( B = T(i) \setminus \{i\} \). Then \( i \not\in T(B) \)
and by induction \( i \not\in T^m(B) \) for all \( m \). Hence \( B \setminus T^m(B) = \emptyset \) for all \( m > 0 \)
and (II) holds for \( A = \bigcup T^m(C) \), \( C \) any finite subset of \( B \). Thus we may
assume \( m > 1 \). Now there exist distinct \( i_1, \ldots, i_{m-1} \) such that \( i \in T(i_{k-1}), \ i_0 = i, i \in T(i_{m-1}) \). The distinctness of the \( i_j \) follows from the fact that \( m \)
is minimal. Let \( I = \{i, i_1, \ldots, i_{m-1}\} \), and \( B = T(I) \setminus I \). Note that by assump-
tion \( B \neq \emptyset \). Now \( I \cap C = \emptyset \) implies \( I \cap T(C) = \emptyset \) for any set \( C \). But
\( B \cap I = \emptyset \). Hence \( T^n(B) \cap I = \emptyset \) for all \( n > 0 \). But then \( B \cap T^n(B) = \emptyset \)
for all \( n > 0 \) and again (II) holds. □

**Bibliography**


**Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27650**