AN ABEL-TAUBER THEOREM ON CONVOLUTIONS WITH THE MÖBIUS FUNCTION

J. L. GELUK

Abstract. Suppose \( n: \mathbb{R}^+ \to \mathbb{R}^+ \) and \( n(x)/x \) is integrable on \((0, \infty)\). For \( s > 0 \) we define

\[
\tilde{n}(s) = s \int_0^\infty \frac{e^{-ux}}{1 - e^{-ux}} n(u) \, du.
\]

In this paper an Abel-Tauber theorem is proved concerning this transform. Moreover the relation between \( \tilde{n}(s) \) and \( \sum_{m \leq s} n(s/m)/m \) is studied.

Introduction. Following earlier work by Landau [10] and others, Ingham proved (in [7]) the following theorem:

THEOREM A. Suppose that

(i) \( f(x) \) is positive and nondecreasing for \( x > 1 \),

(ii) \( F(x) \equiv \sum_{n \leq x} f(x/n) = ax \log x + bx + o(x) \) (\( x \to \infty \)), where \( a \) and \( b \)

are constants. Then

(a) \( f(x) \sim ax (x \to \infty) \),

(b) \( \int_1^\infty (f(x) - ax)/x^2 \, dx = b - ay \) where \( y \) is Euler's constant.

Moreover, for any function \( f(x) \), bounded and integrable over every finite interval \((1, X)\) hypothesis (ii) implies conclusion (b). More recently Jukes [8] extended results of Segal (see [13], [14]) and proved the following theorem which can be considered as a generalization of Theorem A. For a proof of this theorem see also [15].

THEOREM B. Let \( f(x) \) be bounded and integrable on every finite subinterval of \([1, \infty)\) and suppose that

\[
\sum_{n \leq x} f\left( \frac{x}{n} \right) = xg(x) + o\left(x^2 g'(x)\right) \quad (x \to \infty)
\]

where \( g(x) \in C^2[1, \infty) \) is positive and satisfies

(i) \( g'(x) > 0 \) for all \( x \in [1, \infty) \),

(ii) for some real \( r \), \( xg'(x)(\log x)^{-r} \) is nonincreasing for \( x > x_0 \),

(iii) for some real \( s \), \( xg'(x)(\log x)^s \equiv q(x) \) is nondecreasing for \( x > x_1 \) and \( q(x) \to \infty \) (\( x \to \infty \)),

Then \( \int_1^\infty f(t)/t^2 \, dt = g(x) - \gamma xg'(x) + o(xg'(x)) \) (\( x \to \infty \)).

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If we suppose \( f(x) \) to be positive and nondecreasing we also have \( f(x) \sim x^2 g'(x) \) (\( x \to \infty \)).

In this note we consider the situation in Theorem B where \( g(x) \) is allowed to grow faster than in (ii).

We use the concept of regular variation as introduced by Karamata to formulate natural conditions on \( g(x) \) and prove a theorem similar to Theorem B. Moreover, an application is given which gives a second-order condition in a relation considered by Parameswaran in [11]. See our Corollary 1 and the remark after Theorem 4. Parameswaran proves the following theorem [11, Theorem II].

**Theorem C.** If \( \int_0^R n(u)/u \, du \) exists in the Lebesgue sense for every positive \( R \)

\[
f(s) \equiv \exp\left\{ s \int_0^{\infty} \frac{e^{-su}}{1 - e^{-su}} n(u) \, du \right\}
\]

for all positive \( s \) and \( n(u) \) nondecreasing, then the relation \( \log f(s) \sim \int_0^{s^{1/2}} L(u)/u \, du \) as \( s \to 0^+ \) implies the relation \( n(u) \sim L(u) \) as \( u \to \infty \), provided \( L(x) \) is a slowly oscillating function defined for \( x > a \) such that \( L(x) \sim K \exp\left(\int_0^x \delta(u)/u \, du\right) \) as \( x \to \infty \) where \( \delta \) is a nonincreasing function and \( K \) is a positive real number.

We use our Theorem 3 to get a theorem similar to Theorem C.

**RESULTS.** A real valued function \( q \) is said to be regularly varying at infinity if it is positive and measurable on \([A, \infty)\) for some \( A > 0 \) and if for each \( \lambda > 0 \), \( \lim_{x \to \infty} q(\lambda x)/q(x) = \lambda^p \) for some \( \rho \in (-\infty, \infty) \). \( \rho \) is called the exponent of regular variation. Functions \( q \) which satisfy the limit relation with \( \rho = 0 \) are called slowly varying (slowly oscillating).

For an extensive treatment of regular variation the reader is referred to Seneta [16].

Now we define the subclass of slowly varying functions we want to consider.

**Definition.** The nondecreasing function \( q: R^+ \to R \) belongs to the class \( \Pi \) if there exists a slowly varying function \( L \) such that \( q(x) = \int L(t)/t \, dt + L(x) \).

It can be shown that the class \( \Pi \) consists of all nondecreasing functions \( q: R^+ \to R \) for which there exist functions \( a: R^+ \to R^+ \) and \( b: R^+ \to R \) such that for all positive \( x \)

\[
\lim_{t \to \infty} \frac{q(tx) - b(t)}{a(t)} = \log x.
\]

If the last relation holds, then it is true with \( b(t) = q(t) \) and \( a(t) = q(te) - q(t) \). The function \( a(t) \) is of course determined up to asymptotic equivalence and is called the auxiliary function. As a consequence of these facts we mention the following result: if \( q(x) \in \Pi \) with auxiliary function \( L(x) \), \( q_1(x) \) is nondecreasing and \( (q(x) - q_1(x))/L(x) \to C \) (\( x \to \infty \)), where \( C \in R \) is a
constant, then \( q(t) \in \Pi \) with auxiliary function \( L(x) \). It is easy to see that if \( L(x) \) slowly varying, then \( \int_1^\infty L(t)/t \, dt \) is an element of \( \Pi \) with auxiliary function \( L(x) \).

The class \( \Pi \) is a subclass of the slowly varying functions at infinity. Moreover, for each function \( q \in \Pi \) it is possible to find a function \( L_\ast \) such that \( q(x) = \int_1^\infty L_\ast(t)/t \, dt + o(L(x)) \) \((x \to \infty)\) where \( L_\ast \sim L \) is slowly varying.

For the proofs of these statements and further properties see [3], [4] and [5].

We start with two Abelian results.

**Theorem 1.** Suppose \( n: \mathbb{R}^+ \to \mathbb{R}^+ \) is slowly varying, \( n(x)/x \) is (Lebesgue) integrable on finite subintervals of \((0, \infty)\). Then \( \tilde{n}(1/s) \in \Pi \) with auxiliary function \( n(s) \), where \( \tilde{n} \) is defined by

\[
\tilde{n}(s) = s \int_0^\infty \frac{e^{-us}}{1 - e^{-us}} \, n(u) \, du.
\]

Moreover,

\[
\frac{\tilde{n}(1/s) - \int_0^s \frac{n(t)}{t} \, dt}{n(s)} \to 0 \quad (s \to \infty).
\]

**Proof.**

\[
\int_0^s \frac{n(t)}{t} \, dt - \tilde{n}(1/s) = \int_0^1 \left( \frac{1 - e^{-u}}{u} \right) n(su) \, du \quad (s \to \infty).
\]

We have \( \int_0^1 n(su)/n(s) \, du = \int_0^1 n(v) \, dv / sn(s) \to 1 \) \((s \to \infty)\) by Karamata’s theorem on regularly varying functions and since \( 1/u - e^{-u}/(1 - e^{-u}) \) is bounded on \((0, 1)\) we can apply Pratt’s lemma (see [12]).

For the second part we have \( (su)^{-\epsilon}n(su)/s^{-\epsilon}n(s) \to u^{-\epsilon} \) \((s \to \infty)\) uniform on \((1, \infty)\) by Corollary 1.2.1.4 of [3]. So we have

\[
\int_0^s \frac{n(t)}{t} \, dt - \tilde{n}(1/s) \to \int_0^1 \left( \frac{1 - e^{-u}}{u} \right) n(su) \, du - \int_1^\infty \frac{e^{-u}}{1 - e^{-u}} \, du \quad (s \to \infty).
\]

The right side is zero, as is shown by elementary integration. This implies that \( \tilde{n}(1/s) \in \Pi \) with auxiliary function \( n(s) \), since \( \int_0^\infty n(t)/t \, dt \in \Pi \) with auxiliary function \( n(s) \) and \( \tilde{n}(1/s) \) is nondecreasing.

**Remarks.** 1. The case \( n \in RV_{(\alpha)}^\infty (\alpha > 0) \) (regularly varying at infinity with exponent \( \alpha \)) gives analogously \( \tilde{n}(1/s)/n(s) \to \xi(\alpha + 1)\Gamma(\alpha + 1) \) \((s \to \infty)\). See [9, Theorem 1].
The statement of the theorem implies $\tilde{\nu}(1/s) \sim \int_0^s n(t)/t \, dt$ since $n(s) = o(\int_0^s n(t)/t \, dt)$ by Karamata's theorem. See Theorem I of [11].

**Theorem 2.** If we define $\Psi(s) = \sum_{m<s} n(s/m)/m$ where $n$ satisfies the conditions of Theorem 1 and $x n(x)$ is of bounded variation on intervals of the form $(1, x_0)$, then

$$
\left( \Psi(s) - \int_1^s \frac{n(t)}{t} \, dt \right) / n(s) \to \gamma \quad (s \to \infty).
$$

**Proof.** We give the proof by an Euler-Maclaurin kind of argument

$$
\int_{r-1}^r \frac{x}{t} \, n\left( \frac{x}{t} \right) \, dt = x n\left( \frac{x}{r-1} \right) - \int_{r-1}^r \frac{t}{t-1} \, n\left( \frac{x}{t} \right) = \frac{x}{r} n\left( \frac{x}{r} \right) - \int_{r-1}^r \{t\} \frac{x}{t} \, n\left( \frac{x}{t} \right)
$$

where we use the notation $\{t\} = t - \lfloor t \rfloor$.

Summing over $r$ gives

$$
\sum_{1 < r < x} \frac{1}{r} n\left( \frac{x}{r} \right) - \int_1^x \frac{1}{t} n\left( \frac{x}{t} \right) \, dt = n(x) - \int_{\lfloor x \rfloor}^x \frac{1}{t} n\left( \frac{x}{t} \right) \, dt + \int_{\lfloor x \rfloor}^x \{t\} \frac{x}{t} \, n\left( \frac{x}{t} \right).
$$

Now with $M = \sup_{x \in (1, 2)} n(x)$, we have

$$
\left| \int_{\lfloor x \rfloor}^x \frac{1}{t} n\left( \frac{x}{t} \right) \, dt / n(x) \right| \leq \left( M \log \frac{x}{\lfloor x \rfloor} \right) / n(x) < \frac{M}{n(x)(x-1)} \to 0 \quad (x \to \infty)
$$

since $(x-1)n(x)$ is $1$-varying at infinity. Furthermore

$$
\int_{\lfloor x \rfloor}^x \{t\} \frac{x}{t} \, n\left( \frac{x}{t} \right) \, dt = -\frac{1}{x} \int_{\lfloor x \rfloor}^x \frac{x}{t} \, n\left( \frac{x}{t} \right) \, dt \sim -n(x) \int_{\lfloor x \rfloor}^x \frac{1}{t} \, dt \sim (\gamma - 1)n(x)
$$

since

$$
\left( \frac{x}{t} n\left( \frac{x}{t} \right) \right) / x n(x) \to t^{-1}
$$

uniform for $t \in (1, \infty)$ (see [3, Corollary 1.2.1.4]) and $\int_1^\infty (t) / t^2 \, dt = 1 - \gamma$ (see e.g. [6]). This proves Theorem 2.

**Remarks.** 1. The theorem implies $(\Psi(tx) - \Psi(t))/\Psi(te) \to \log x$ $(t \to \infty)$ where $x > 0$ is fixed. If $\Psi$ is nondecreasing we have $\Psi \in \Pi$.

2. With the additional supposition that $\Psi$ is nondecreasing, a simpler proof is possible:

$$
\tilde{\nu}(s) - s \int_0^1 \frac{e^{-us}}{1 - e^{-us}} \, n(u) \, du = s \int_1^\infty \sum_{k=1}^\infty e^{-ku} n(u) \, du = \int_0^\infty e^{-us} \, d\Psi(u).
$$
Since
\[ \frac{1}{s} \int_0^1 \frac{e^{-u/s}}{1 - e^{-u/s}} n(u) \, du - \int_0^1 \frac{n(u)}{u} \, du = o(n(s)) \quad (s \to \infty) \]
as in Theorem 1 we have
\[ \left( \frac{1}{s} - \int_0^1 \frac{n(u)}{u} \, du - n(s) \right) / n(s) \to -\gamma \quad (s \to \infty) \]
(see [5, Theorem 1]). Combining this with Theorem 1 gives the desired result.

3. The case \( n \in RV^o_a (\alpha > 0) \) in Remark 2 gives \( \Psi(s)/(\int_0^s n(t)/t \, dt) \to \alpha \xi(\alpha + 1) (s \to \infty) \).

4. The statement of the theorem implies \( \Psi(s) \sim \int_0^s n(t)/t \, dt \) since \( n(s) = o(\int_0^s n(t)/t \, dt) \) \( (s \to \infty) \).

As a consequence of Remark 2 we mention the following.

**Corollary 1.** Suppose \( n : R^+ \to R^+ \) satisfies the conditions \( \int_0^s n(u)/u \, du < \infty \) for \( R > 0 \) and \( \sum_{m \leq s} n(s/m)/m \) is nondecreasing for \( s > 0 \). The assertions \( \sum_{m \leq s} n(s/m)/m \in \Pi \) with auxiliary function \( L(s) \) and \( \tilde{n}(1/s) \in \Pi \) with auxiliary function \( L(s) \) are equivalent. Both imply
\[ \left( \sum_{m \leq s} \frac{1}{m} n\left( \frac{s}{m} \right) - \tilde{n}(1/s) + \int_0^1 \frac{n(t)}{t} \, dt \right) / L(s) \to \gamma \quad (s \to \infty). \]

For the Tauberian counterpart of Theorems 1 and 2 we need three lemmas.

**Lemma 1.** If
\begin{itemize}
  \item[(i)] \( L(x) \) is nondecreasing for \( x > 0 \), slowly varying, \( L(x) \to \infty \) \( (x \to \infty) \),
  \item[(ii)] \( L(x)/L(x - 1) < 1 + x^{-\alpha} \) for some \( \alpha > 0 \), \( x > x_0 = x_0(\alpha) \), then \( \sum_{m \leq x} \mu(m)/m \) \( L(x/m) = o(L(x)) \) \( (x \to \infty) \).
\end{itemize}

**Proof.** We define \( a_n = L(n) - L(n - 1) \) for \( n > 2 \), \( a_1 = L(1) \). Then
\[ \sum_{m \leq x} \frac{\mu(m)}{m} L\left( \left\lfloor \frac{x}{m} \right\rfloor \right) = \sum_{m \leq x} \frac{\mu(m)}{m} \sum_{\nu \leq x/m} a_x = \sum_{m \leq x} a_m N\left( \frac{x}{m} \right) \]
where \( N(x) = \sum_{m \leq x} \mu(m)/m \). Since \( |N(x)| < \epsilon \) for \( x > x_\epsilon \) (see [10]) we have
\[ \left| \sum_{m \leq x/xi} a_m N\left( \frac{x}{m} \right) \right| < \epsilon \sum_{m \leq x/xi} a_m = \epsilon L\left( \left\lfloor \frac{x}{x_\epsilon} \right\rfloor \right) < \epsilon(1 + \epsilon)L(x) \]
\( (x > x_\epsilon) \)
and
\[ \left| \sum_{x/x_\epsilon < m \leq x} a_m N\left( \frac{x}{m} \right) \right| < c \sum_{x/x_\epsilon < m \leq x} a_m < c \left( L(x) - L\left( \frac{x}{x_\epsilon} \right) \right) = o(L(x)) \]
\( (x \to \infty). \)

For \( x > x_0(\alpha) \) we have by (ii), \( L(x) - L([x]) < L(x) - L(x - 1) < L(x)/x^\alpha \) since \( L \) is nondecreasing. Hence,
\[
\left| \sum_{1 \leq m < x/x_0} \frac{\mu(m)}{m} \left( L\left( \frac{x}{m} \right) - L\left( \left\lfloor \frac{x}{m} \right\rfloor \right) \right) \right| \leq \sum_{m \leq x/x_0} \frac{1}{m} L\left( \frac{x}{m} \right) x^{a/m^a} \\
\leq x^{-\alpha} \left\{ \int_{1}^{x/x_0} \frac{1}{u^{1-a}} L\left( \frac{x}{u} \right) du + L(x) \right\} \\
= \int_{x_0}^{x} \frac{L(v)}{v^{1+a}} \, dv + o(L(x)) = o(L(x)),
\]
if we choose \(0 < \alpha < 1\).

We estimate the last sum as follows.
\[
\left| \sum_{x/x_0 < m \leq x} \frac{\mu(m)}{m} \left( L\left( \frac{x}{m} \right) - L\left( \left\lfloor \frac{x}{m} \right\rfloor \right) \right) \right| \\
\leq 2 \sum_{x/x_0 < m \leq x} \frac{1}{m} L\left( \frac{x}{m} \right) < 2L(x_0) \sum_{x/x_0 < m \leq x} \frac{1}{m} \\
= O(1) = o(L(x)) \quad (x \to \infty).
\]
This proves the lemma.

**Remark.** The conclusion of the lemma is incorrect for arbitrary slowly varying functions.

Recently Erdős and Segal constructed a function (see [1]) such that
\[
2m^{1+\alpha} = o(L(x)) \quad (x \to \infty).
\]

**Lemma 2.** Suppose \( \sum_{m \leq x} (\mu(m)/m) L(x/m) = f(x) \) where \( L \) satisfies the conditions of Lemma 1. Then \( n(x) \sim L(x) \) \( (x \to \infty) \).

**Proof.** Möbius inversion and Lemma 1 give
\[
n(x) = \sum_{m \leq x} \frac{\mu(m)}{m} \int_{1}^{x/m} \frac{L(u)}{u} \, du + \sum_{m \leq x} \frac{\mu(m)}{m} L\left( \frac{x}{m} \right) \\
= \int_{1}^{x} \frac{L(u)}{u} N\left( \frac{x}{u} \right) \, du + o(L(x)) \\
= \int_{1}^{x} L\left( \frac{x}{v} \right) \frac{N(v)}{v} \, dv + o(L(x)) \sim L(x) \int_{1}^{\infty} \frac{N(v)}{v} \, dv \sim L(x) \\
\quad (x \to \infty)
\]
by dominated convergence, since
\[
\int_{1}^{x} \frac{1}{v} \sum_{k \leq v} \frac{\mu(k)}{k} \, dv = \sum_{k \leq x} \frac{\mu(k)}{k} \int_{k}^{x} \frac{dv}{v} \\
= \log x \sum_{k \leq x} \frac{\mu(k)}{k} - \sum_{k \leq x} \frac{\mu(k)}{k} \log k \to 1 \quad (x \to \infty)
\]
(see [10]).

**Remark.** The proof of Lemma 2 implies the following. If \( h(x) = \int_{1}^{x} L(u)/u \, du + L(x) \) where \( L \) satisfies the conditions of Lemma 1, then
\[
\sum_{m \leq x} (\mu(m)/m) h(x/m) \sim (\int_{1}^{1} s \, dh(s)) / x,
\]
which is the auxiliary function of \( h(x) \). For functions \( h(x) \) of the form \( \int_{1}^{x} L(u)/u \, du \), \( L \) slowly varying the
conditions of Lemma 1 are not necessary for this result. Compare with Theorem 1 of [13].

The following lemma gives the same statement under different conditions on \( L(x) \).

**Lemma 3.** Suppose \( \sum_{m \leq x} \frac{n(x/m)}{m} = \int_1^x L(u)/u \, du + L(x) \) where \( L(x) \to \infty \) and \( L(x) \in \Pi \). Then \( n(x) \sim L(x) \) \((x \to \infty)\).

**Proof.** We write \( L(x) = \int_1^x \gamma(u)/u \, du + o(\gamma(x)) \) where \( \gamma \) is slowly varying. Then Möbius inversion gives

\[
n(x) = \sum_{m \leq x} \frac{\mu(m)}{m} \int_1^x \frac{L(u) + \gamma(u)}{u} \, du + \sum_{m \leq x} \frac{\mu(m)}{m} o(\gamma(x/m)).
\]

The first part is asymptotic to \( L(x) \) as in Lemma 2 since \( \gamma(x) = o(L(x)) \) \((x \to \infty)\). For the second term we proceed as follows. Suppose \(|o(\gamma(x))| < \varepsilon \gamma(x) \) for \( x > x_0 \). Then we have

\[
\left| \sum_{m \leq x} \frac{\mu(m)}{m} o(\gamma(x/m)) \right| \leq \varepsilon \sum_{m \leq x} \frac{1}{m} \gamma(x/m) + \sum_{m \leq x} \frac{1}{m} |o(\gamma(x/m))| < \varepsilon \sum_{x_0 < m \leq x} \frac{1}{m} \gamma(x/m)/m + O(1)
\]

where \( c = \sup_{x \in (1, x_0]} |o(\gamma(x))| \). Now we have \( \sum_{m \leq x} \gamma(x/m)/m \sim L(x) \) \((x \to \infty)\) by Theorem 2. This proves the lemma.

**Theorem 3.** Suppose

(i) \( \int_1^x L(u)/u \, du + o(L(x)) = \sum_{m \leq x} n(x/m)/m, \) where \( L(x) \to \infty \) \((x \to \infty)\) is slowly varying,

(ii) \( R(x) \equiv \sum_{m \leq x} n(x/m)/m \) is nondecreasing,

(iii) \( L_*(x) \equiv (\int_1^x dt \, dR(t))/x \) is nondecreasing,

(iv) \( L_*(x)/L_*(x-1) < 1 + x^{-\alpha} \) with \( \alpha > 0 \) for \( x > x_0(\alpha) \), then \( n(x) \sim L(x) \) \((x \to \infty)\).

**Proof.** We have \( R(x) = \int_1^x L_*(t)/t \, dt + L_*(x) \) where \( L_*(x) \) is defined as above and \( L_*(x) \sim L(x) \) \((x \to \infty)\) satisfies the conditions of Lemma 2. This proves the theorem.

**Corollary 2.** Combination with Theorem 2 yields that with the assumptions of Theorem 3 and \( n(x)/x \) integrable we have

\[
\int_1^x \frac{n(t)}{t} \, dt = \int_1^x \frac{L(u)}{u} \, du - \gamma L(x) + o(L(x)) \quad (x \to \infty).
\]

This is the same conclusion as in the papers of Segal and Jukes.

**Theorem 4.** Suppose (i) and (ii) of Theorem 3 and \( L_*(x) \in \Pi \). Then \( L(x) \sim n(x) \) \((x \to \infty)\).

**Proof.** We use now Lemma 3. The rest of the proof is as in Theorem 3.

Combining Corollary 1 with Theorems 3 or 4 gives the following Tauberian theorem.
THEOREM 5. If \( \tilde{n}(1/s) = \int_1^s L(u)/u \, du + o(L(s)) \) (\( s \to \infty \)) with \( L(s) \to \infty \) slowly varying and \( n: R^+ \to R^+ \) satisfies the conditions

(i) \( n(u)/u \) is integrable on \( (0, R) \) for every \( R > 0 \),

(ii) \( R(x) = \sum_{m<x} n(x/m)/m \) is nondecreasing,

(iii) \( L_m(x) = (\int_0^x dR(t))/x \) is nondecreasing,

and

(iv) \( L_\alpha(x) \in \Pi \),

or

(v) \( L_\alpha(x)/L_\alpha(x-1) < 1 + x^{-\alpha} \) with \( \alpha > 0 \) for \( x > x_0(\alpha) \),

then

\[
\frac{\int_1^s n(t) \, dt - \int_1^s L(t) \, dt}{L(s)} \to 0 \quad (s \to \infty).
\]

PROOF. Applying Corollary 1 and Theorems 3 or 4 we have \( n(x) \sim L(x) \). Application of Theorem 1 yields the result.

For regularly varying functions the following analogue of Theorem 5 is well known.

THEOREM 6. If \( \tilde{n}(1/s) \sim \zeta(\alpha + 1) \Gamma(\alpha + 1)s^\alpha L(s) \) where \( L(s) \in RV^\alpha \) with \( \alpha > 0 \) and \( n(u) \) satisfies the following conditions

(i) \( n(u)/u \) and \( (n(u)/u) \log u \) are integrable on \( (0, R) \) for every \( R > 0 \),

(ii) \( n(u) \) is nondecreasing,

then \( n(u) \sim u^\alpha L(u) \) (\( u \to \infty \)).

For a proof see [9, Theorem 2].

EXAMPLE 1. The function \( g(x) = e^{\log x} \) belongs to the class \( \Pi \). This function does not satisfy the assumptions of Theorem B since the auxiliary function \( xg'(x) \) is growing too fast. With the given conditions however we can apply Corollary 2 to get the desired result. On the other hand, for functions which belong to the class \( \Pi \) with a bounded auxiliary function our Tauberian results are not applicable.

2. The conditions on \( L(x) \) in Lemmas 2 and 3 do not imply each other. This is shown by the following examples. If \( L_1(x) = 2 \log x + \sin \log x \), and \( L_2(x) = [\log x] + \log^2(x + 1) \), then \( L_1 \) (resp., \( L_2 \)) satisfies the conditions of Lemma 2 (resp., 3) and not the conditions of Lemma 3 (resp., 2).

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