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MODIFIED POISSON KERNELS ON RANK ONE SYMMETRIC SPACES

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ABSTRACT. An extension is obtained to the case of a real rank one noncompact symmetric space G/K of the solution of the following problem on half-spaces: given an arbitrary continuous function $f(x)$ on $\mathbf{R}^n$, is it possible to find a function F on $\mathbf{R}^n \times \mathbf{R}^+$ such that $F(x, y)$ is continuous for $y > 0$, harmonic for $y > 0$ and such that $F(x, 0) = f(x)$?

1. Introduction and notations. Let G be a connected noncompact semisimple Lie group with finite center and of real rank one. Let θ be a Cartan involution of the Lie algebra $\mathfrak{g}$ of G corresponding to the Cartan decomposition $\mathfrak{g} = k \oplus \mathfrak{p}$. Let K be the corresponding maximal compact subgroup of G and G/K the associated symmetric space. Let $G = VAK$ be an Iwasawa decomposition of G, so every element $g \in G$ can be written as $g = va_tk$, where $v \in V$, $a_t = \exp tH \in A$ (the element H in $\mathfrak{a}$–the Lie algebra of A–is chosen in such a way that $\alpha(H) = 1$) and $k \in K$. We also denote by $H(g)$ the logarithm of the A-component a_t of g in $\mathfrak{a}$. We can write every element of V as $v = \exp X \exp Y$ with $X \in \mathfrak{g}_{-\alpha}$, $Y \in \mathfrak{g}_{-2\alpha}$ where α and 2α (or α) denote the positive restricted roots (or root) of $(\mathfrak{g}, \mathfrak{a})$. The (real) dimensions of the root spaces $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$ are denoted, respectively, by p and q. If B is the Killing form of $\mathfrak{g}$, we put $\|Y\|^2 = -B(Y, \theta Y)$ for $Y \in \mathfrak{g}$. We also put $2\rho = (p + 2q)\alpha$. For M the centralizer of A in K, the Poisson kernel P is defined on $G/K \times K/M$ (or on $G/K \times B$ for some boundary B isomorphic to K/M; cf. Koranyi [5]) by

$$P(gK, kM) = \exp(-2\rho H(g^{-1}k)).$$

Making use of the formulas on pp. 65 and 67 of Helgason's paper [3] we can consider P as a function defined on $G/K \times V$ by the following expression:

$$P(a_tK, v_0) = \exp(2\rho \log a_t)\exp(-2\rho H(a_{-t}v_0a_t)). \tag{1}$$

Here, the Poisson kernel is always given by (1).

A C^∞ function f on G/K is called harmonic if it is annihilated by all left-invariant differential operators on G/K without constant term. In particular, the Poisson kernel P is harmonic on G/K for each fixed element of V. It is also known that, if f is a bounded continuous function on V and F is its Poisson integral, namely,

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$$F(gK) = \int_V P(gK, v_0) f(v_0)\, dv_0,$$

where dv_0 is the appropriate Haar measure on V, then F is harmonic on G/K and $\lim_{t\to\infty} F(va_tK) = f(v)$. If we think of G/K as a generalization of the upper half-plane and its boundary V as a generalization of the real line, this limit generalizes vertical limits in the upper half-plane. For all these definitions and notations we refer to Helgason [**2**].

The following theorem generalizes to rank one symmetric spaces the result obtained by Finkelstein and Scheinberg for $\mathbf{R}^n$ [**1**].

THEOREM. *Let f be an arbitrary continuous function on V. Then there exists a kernel W_f defined on $G/K \times V$ and depending on f such that*:

(i) $\int_V W_f(va_tK, v_0) f(v_0) dv_0 = F(va_tK)$ *converges for every* $va_tK \in G/K$.

(ii) $\lim_{t\to\infty} F(va_tK) = f(v)$, *for every* $va_tK \in G/K$.

(iii) *F is a harmonic function on G/K.*

REMARK. The theorem is still true if we replace the condition "f continuous" by the condition "f locally integrable".

2. The Poisson kernel. The Poisson kernel P on $G/K \times V$ is in the rank one case, given by the formula (see [3, pp. 65–67])

$$P(a_tK, v_0) = \left[\frac{e^{2t}}{\left(1 + c\|e^t X_0\|^2\right)^2 + 4c\|e^{2t} Y_0\|^2}\right]^{p/2+q}$$

where $v_0 = \exp X_0 \exp Y_0$ with $X_0 \in \mathfrak{g}_{-\alpha}$, $Y_0 \in \mathfrak{g}_{-2\alpha}$ and where $c = 1/4(p + 4q)$. If $v = \exp X \exp Y$ with $X \in \mathfrak{g}_{-\alpha}$, $Y \in \mathfrak{g}_{-2\alpha}$, then

$$v^{-1}v_0 = \exp(X_0 - X)\exp(Y_0 - Y + \{X, X_0\}),$$

where $\{\ ,\ \}$ denotes a bilinear form from $\mathfrak{g}_{-\alpha} \times \mathfrak{g}_{-\alpha}$ into $\mathfrak{g}_{-2\alpha}$. So, from the identity $P(va_tK, v_0) = P(a_tK, v^{-1}v_0)$, we get

$$P(va_tK, v_0) = \left[\frac{e^{2t}}{\left(1 + ce^{2t}\|X_0 - X\|^2\right)^2 + 4ce^{4t}\|Y_0 - Y + \{X, X_0\}\|^2}\right]^{p/2+q}$$

We put, for notational convenience, $|v_0| = c^2\|X_0\|^4 + 4c\|Y_0\|^2$, and $\sigma = -(p/2 + q)$.

Let us define

$$Z_1 = Z_1(v, v_0) = 2c\left[\|X_0\|^2 + \|X\|^2 + 2B(X_0, \theta X)\right]$$

and

$$\begin{aligned} Z_2 = Z_2(v, v_0) = c^2\big[&\|X\|^4 + 2\|X_0\|^2\|X\|^2 + 4B^2(X_0, \theta X) \\ &+ 4\|X_0\|^2 B(X_0, \theta X) + 4\|X\|^2 B(X_0, \theta X)\big] \\ +4c\big[&\|Y\|^2 + \|\{X, X_0\}\|^2 + 2B(Y_0, \theta Y) \\ &- 2B(Y_0, \theta\{X, X_0\}) + 2B(Y, \theta\{X, X_0\})\big]. \end{aligned}$$

We may now write

$$P(va_tK, v_0) = \left(e^{2t}|v_0|\right)^{\sigma}(1 + Z)^{\sigma},$$

where

$$Z = \frac{1 + e^{2t}Z_1 + e^{4t}Z_2}{e^{4t}|v_0|}, \tag{2}$$

is a function on $G/K \times V$. Finally, we call $\tilde{Z}$ the expression that we obtain in (2) if we take the absolute value of each term making up Z_1 and Z_2.

Let va_tK be a fixed element in G/K. Since $\tilde{Z}$ decreases to zero when $|v_0|$ tends to infinity, it is possible to choose a least positive constant C_1 (depending on va_tK) in such a way that, for every $v_0 \in V$ with $|v_0| > C_1$, then $\tilde{Z} < \frac{1}{2}$. Obviously, for the same v_0's, also the absolute value of Z is less than or equal to $\frac{1}{2}$. Thus, for every va_tK fixed, the following identity

$$P(va_tK, v_0) = \left(e^{2t}|v_0|\right)^{\sigma} \sum_{j>0} (-1)^j b_j Z^j,$$

where

$$b_j = \frac{(-\sigma + j - 1)!}{(-\sigma - 1)!j!},$$

is satisfied for all v_0 such that $|v_0| > C_1$.

REMARKS. (1) Let $v \in V$ be a fixed element and let t belong to the interval $[0, \infty)$. Since the constant C_1, depending on va_tK, decreases when t goes to infinity, there exists a least positive constant C_2, depending only on v, such that if $v_0 \in V$ and $|v_0| > C_2$, then $\tilde{Z} < \frac{1}{2}$ for every positive t.

(2) If H is a compact set in G/K, it is possible to choose a positive constant C_3, depending on H, in such a way that for all $v_0 \in V$ with $|v_0| > C_3$, then $\tilde{Z} < \frac{1}{2}$ for every $va_tK \in H$.

(3) Obviously the series $\Sigma_{j>0}(-1)^j b_j Z^j$ converges where the series $\Sigma_{j>0} b_j \tilde{Z}^j$ converges. Moreover, the series $\Sigma_{j>0} b_j 2^{-j}$ is convergent.

In order to reorder the series expression for the Poisson kernel P, we consider the action of A on V defined by

$$a_s \cdot v_0 = \exp(e^s X_0)\exp(e^{2s} Y_0),$$

where $v_0 = \exp X_0 \exp Y_0$. We assume also $s > 0$. The expression of P as a function on $G/K \times (A \cdot V)$ is then the following:

$$P(va_tK, a_s \cdot v_0)$$

$$= \left[\frac{e^{2t}}{\left(1 + ce^{2t}\|e^s X_0 - X\|^2\right)^2 + 4ce^{4t}\|e^{2s}Y_0 - Y + e^s\{X, X_0\}\|^2} \right]^{-\sigma}.$$

Let us define

$$R_1 = R_1(v, a_s \cdot v_0) = 2c\left[e^{2s}\|X_0\|^2 + \|X\|^2 + 2e^s B(X_0, \theta X)\right]$$

and

$$R_2 = R_2(v, a_s \cdot v_0) = c^2\big[\|X\|^4 + 2e^{2s}\|X_0\|^2\|X\|^2 + 4e^{2s}B^2(X_0, \theta X) + 4e^{3s}\|X_0\|^2 B(X_0, \theta X) + 4e^{s}\|X\|^2 B(X_0, \theta X)\big]$$
$$+4c\big[\|Y\|^2 + e^{2s}\|\{X, X_0\}\|^2 + 2e^{2s}B(Y_0, \theta Y) - 2e^{3s}B(Y_0, \theta\{X, X_0\}) + 2e^{s}B(Y, \theta\{X, X_0\})\big].$$

So we can express

$$P(va_tK, a_s \cdot v_0) = \left(e^{2(t+2s)}|v_0|\right)^\sigma[1 + R]^\sigma = \left(e^{2(t+2s)}|v_0|\right)^\sigma \sum_{j>0} (-1)^j b_j R^j,$$

where

$$R = \frac{1 + e^{2t}R_1 + e^{4t}R_2}{e^{4(t+s)}|v_0|},$$

is a function on $G/K \times (A \cdot V)$. Collecting together the elements of the sum with the same homogeneity in e^s, we get

$$P(va_tK, a_s \cdot v_0) = \left(e^{2(t+2s)}|v_0|\right)^\sigma \sum_{j>0} h_j(va_tK, v_0)e^{-js} = \left(e^{2t}|v_0|\right)^\sigma \sum_{j>0} h_j(va_tK, v_0)e^{(4\sigma-j)s},$$

where the functions h_j are defined in such a way that the identity is true when the series is convergent. We define also

$$H_j(va_tK, v_0) = \left(e^{2t}|v_0|\right)^\sigma h_j(va_tK, v_0).$$

If, in particular, $s = 0$ we have

$$P(va_tK, v_0) = \sum_{j>0} H_j(va_tK, v_0).$$

3. The W_f kernel. Let H_j: $G/K \times V \to \mathbf{R}$ be the functions defined in §2. They are piecewise continuous and bounded on V. Now we want to prove that they are also harmonic as functions on G/K. Let v_0 be a fixed element in V. Then, for every va_tK in G/K we choose a positive real number s large enough so that the following identity holds:

$$P(va_tK, a_s \cdot v_0) = \sum_{j>0} H_j(va_tK, v_0)e^{(4\sigma-j)s}.$$

Taking the limit on both sides as s tends to infinity we find

$$H_0(va_tK, v_0) = \lim_{s\to\infty} P(va_tK, a_s \cdot v_0)e^{4\sigma s}.$$

Since for every fixed s the function P is harmonic, $H_0(\cdot, v_0)$ is also harmonic on G/K. In the same way, we obtain, for $j = 1,2, \ldots,$

$$H_j(va_tK, v_0) = \lim_{s\to\infty} e^{js}\left[P(va_tK, a_s\cdot v_0)e^{4\sigma s} - \sum_{r=0}^{j-1} H_r(va_tK, v_0)e^{-rs}\right].$$

The harmonicity of the function $H_j(\cdot, v_0)$ on G/K is thus proved.

Let us consider now the function f we want to extend. Let φ be a positive continuous function from V into $\mathbf{R}$ such that

$$|f(v_0)|\varphi(v_0) < |v_0|^{-2+\sigma/2},$$

for every $v_0 \in V$. Let us define a nonnegative integer valued function J on V by $J(v_0) = J_0$, where J_0 is the least integer for which the following inequality

$$\sum_{j>(J_0-1)/3} b_j 2^{-j} < \varphi(v_1),$$

holds for every $v_1 \in V$ such that $1 < |v_1| < |v_0|$ and where $J_0 = 0$ if $|v_0| < 1$. The function J is well defined and depends on the choice of φ. Moreover, if $v_1, v_2 \in V$ and $|v_1| < |v_2|$, then $J(v_1) < J(v_2)$.

We define now a function Q on $G/K \times V$ by

$$Q(va_tK, v_0) = \sum_{j<J_0} H_j(va_tK, v_0).$$

Finally, we define on $G/K \times V$ the kernel W_f associated to the function f by

$$W_f(va_tK, v_0) = P(va_tK, v_0) - Q(va_tK, v_0).$$

We estimate this kernel for a fixed element va_tK in G/K. Let us suppose $C_1 > 1$. For any $v_0 \in V$ such that $|v_0| > C_1$ we have

$$\begin{aligned}|W_f(va_tK, v_0)| &= \left|\sum_{j>J_0} H_j(va_tK, v_0)\right| \\ &< (e^{2t}|v_0|)^\sigma \sum_{j>\frac{1}{3}(J_0-1)} b_j 2^{-j} < (e^{2t}|v_0|)^\sigma \varphi(v_0).\end{aligned}$$

REMARKS. (1) If $v \in V$ is fixed and a_t varies in A when t belongs to the interval $[0, \infty)$ we have that

$$|W_f(va_tK, v_0)| < (e^{2t}|v_0|)^\sigma \varphi(v_0),$$

for all $v_0 \in V$ such that $|v_0| > C_2$.

(2) Analogously, when va_tK varies in a compact set H of G/K, there exists a positive constant D such that, for every $v_0 \in V$ with $|v_0| > C_3$, we have

$$|W_f(va_tK, v_0)| < D\varphi(v_0).$$

4. Proof of the theorem. Let $T \in \mathbf{R}^+$. We define $E(T) = \{v_0 \in V: |v_0| < T\}$.

(i) Let $va_tK \in G/K$ be a fixed element and suppose $T > \max(1, C_1)$. We can write

$$F(va_tK, v_0) = \int_{E(T)} P(va_tK, v_0)f(v_0)\, dv_0 - \int_{E(T)} Q(va_tK, v_0)f(v_0)\, dv_0$$
$$+ \int_{V\setminus E(T)} W_f(va_tK, v_0)f(v_0)\, dv_0.$$

The first integral is finite because Pf is a bounded continuous function on the compact set $E(T)$. The second integral is also finite because on $E(T)$, Qf is a finite sum of bounded and measurable functions. For the third integral we have

$$\int_{V\setminus E(T)} |W_f(va_tK, v_0)f(v_0)|\, dv_0 < \int_{V\setminus E(T)} (e^{2t}|v_0|)^\sigma \varphi(v_0)|f(v_0)|\, dv_0$$
$$< (e^{2t})^\sigma \int_{V\setminus E(T)} |v_0|^{-2+\sigma/2}\, dv_0 = \frac{e^{2t\sigma}}{T},$$

which is finite. So (i) is proved.

(ii) Fix $v \in V$. Take $T > \max(1, C_2)$ and observe that $T > |v|$. We know that

$$\lim_{t\to\infty} \int_V P(va_tK, v_0)f(v_0)\chi_{E(T)}(v_0)\, dv_0 = f(v),$$

if $\chi_{E(T)}$ is the characteristic function of the compact set $E(T)$. We can estimate

$$\lim_{t\to\infty} \left| \int_{E(T)} Q(va_tK, v_0)f(v_0)\, dv_0 \right|$$
$$< \lim_{t\to\infty} (e^{2t}|v_0|)^\sigma \int_{E(T)} \sum_{j<J_0} |h_j(va_tK, v_0)f(v_0)|\, dv_0 = 0,$$

because $|h_jf|$ are bounded and measurable functions on $E(T)$ and the sum is finite on $E(T)$.

Finally, we estimate

$$\lim_{t\to\infty} \int_{V\setminus E(T)} |W_f(va_tK, v_0)f(v_0)|\, dv_0 < \frac{1}{T} \lim_{t\to\infty} e^{2t\sigma} = 0.$$

So we have proved that $\lim_{t\to\infty} F(va_tK) = f(v)$, for every $va_tK \in G/K$.

(iii) Let H be a fixed compact set in G/K. Let T be any integer such that $T > \max(1, C_3)$. We have

$$\lim_{T\to\infty} \int_{V\setminus E(T)} |W_f(va_tK, v_0)f(v_0)|\, dv_0 < D \lim_{T\to\infty} \frac{1}{T} = 0$$

uniformly on H. So, uniformly on H, we have

$$F(va_tK) = \lim_{T\to\infty} \int_{E(T)} P(va_tK, v_0)f(v_0)\, dv_0$$
$$- \lim_{T\to\infty} \int_{E(T)} Q(va_tK, v_0)f(v_0)\, dv_0. \tag{3}$$

Both functions in (3) are harmonic on G/K, so (iii) is proved.

REMARK. The reader may wish to compare our "explicit" construction with the more general techniques of [**4**].

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