MODIFIED POISSON KERNELS ON RANK ONE
SYMMETRIC SPACES

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Abstract. An extension is obtained to the case of a real rank one noncom-
pact symmetric space $G/K$ of the solution of the following problem on
half-spaces: given an arbitrary continuous function $f(x)$ on $\mathbb{R}^n$, is it possible
to find a function $F$ on $\mathbb{R}^n \times \mathbb{R}^+$ such that $F(x, y)$ is continuous for $y > 0$,
harmonic for $y > 0$ and such that $F(x, 0) = f(x)$?

1. Introduction and notations. Let $G$ be a connected noncompact semisimple
Lie group with finite center and of real rank one. Let $\theta$ be a Cartan involution
of the Lie algebra $\mathfrak{g}$ of $G$ corresponding to the Cartan decomposition
$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $K$ be the corresponding maximal compact subgroup of $G$ and
$G/K$ the associated symmetric space. Let $G = VAK$ be an Iwasawa decom-
position of $G$, so every element $g \in G$ can be written as $g = v a k$, where
$v \in V$, $a = \exp H \in A$ (the element $H$ in $a$–the Lie algebra of $A$–is chosen
in such a way that $a(H) = 1$) and $k \in K$. We also denote by $H(g)$ the
logarithm of the $A$-component $a$, of $g$ in $a$. We can write every element of $V$
as $v = \exp X \exp Y$ with $X \in \mathfrak{g}_{-\alpha}$, $Y \in \mathfrak{g}_{-2\alpha}$ where $\alpha$ and $2\alpha$ (or $\alpha$) denote
the positive restricted roots (or root) of $(\mathfrak{g}, \alpha)$. The (real) dimensions of the
root spaces $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$ are denoted, respectively, by $p$ and $q$. If $B$ is the
Killing form of $\mathfrak{g}$, we put $\| Y \|^2 = - B(Y, \theta Y)$ for $Y \in \mathfrak{g}$. We also put
$2\rho = (p + 2q)\alpha$. For $M$ the centralizer of $A$ in $K$, the Poisson kernel $P$
is defined on $G/K \times K/M$ (or on $G/K \times B$ for some boundary $B$ isomorphic
to $K/M$; cf. Koranyi [5]) by

$$P(gk, kM) = \exp(-2\rho H(g^{-1}k)).$$

Making use of the formulas on pp. 65 and 67 of Helgason’s paper [3] we can
consider $P$ as a function defined on $G/K \times V$ by the following expression:

$$P(aK, v_0) = \exp(2\rho \log a) \exp(-2\rho H(a^{-1}v_0a)).$$

Here, the Poisson kernel is always given by (1).

A $C^\infty$ function $f$ on $G/K$ is called harmonic if it is annihilated by all
left-invariant differential operators on $G/K$ without constant term. In partic-
ular, the Poisson kernel $P$ is harmonic on $G/K$ for each fixed element of $V$. It
is also known that, if $f$ is a bounded continuous function on $V$ and $F$ is its
Poisson integral, namely,
$$F(gK) = \int_V P(gK, v_0)f(v_0) \, dv_0,$$

where $dv_0$ is the appropriate Haar measure on $V$, then $F$ is harmonic on $G/K$ and $\lim_{t \to \infty} F(v_{aK}) = f(v)$. If we think of $G/K$ as a generalization of the upper half-plane and its boundary $V$ as a generalization of the real line, this limit generalizes vertical limits in the upper half-plane. For all these definitions and notations we refer to Helgason [2].

The following theorem generalizes to rank one symmetric spaces the result obtained by Finkelstein and Scheinberg for $\mathbb{R}^n$ [1].

**Theorem.** Let $f$ be an arbitrary continuous function on $V$. Then there exists a kernel $W$, defined on $G/K \times V$ and depending on $f$ such that:

(i) $\int_V W_f(v_{aK}, v_0)f(v_0) \, dv_0 = F(v_{aK})$ converges for every $v_{aK} \in G/K$.

(ii) $\lim_{t \to \infty} F(v_{aK}) = f(v)$, for every $v_{aK} \in G/K$.

(iii) $F$ is a harmonic function on $G/K$.

**Remark.** The theorem is still true if we replace the condition "$f$ continuous" by the condition "$f$ locally integrable".

2. The Poisson kernel. The Poisson kernel $P$ on $G/K \times V$ is in the rank one case, given by the formula (see [3, pp. 65–67])

$$P(a,K, v_0) = \left[ \frac{e^{2t}}{\left(1 + c\|e^X_0\|^2 + 4c\|e^{2t}Y_0\|^2\right)} \right]^{p/2 + q},$$

where $v_0 = \exp X_0 \exp Y_0$ with $X_0 \in \mathfrak{g}_{-a}$, $Y_0 \in \mathfrak{g}_{-2a}$ and where $c = 1/4(p + 4q)$. If $v = \exp X \exp Y$ with $X \in \mathfrak{g}_{-a}$, $Y \in \mathfrak{g}_{-2a}$, then

$$v^{-1}v_0 = \exp(X_0 - X)\exp(Y_0 - Y + \{X, X_0\}),$$

where $\{,\}$ denotes a bilinear form from $\mathfrak{g}_{-a} \times \mathfrak{g}_{-a}$ into $\mathfrak{g}_{-2a}$. So, from the identity $P(v_{aK}, v_0) = P(a,K, v^{-1}v_0),$ we get

$$P(v_{aK}, v_0) = \left[ \frac{e^{2t}}{\left(1 + c\|e^X_0 - X\|^2 + 4c\|e^{2t}Y_0 - Y + \{X, X_0\}\|^2\right)} \right]^{p/2 + q}.$$

We put, for notational convenience, $|v_0| = c^2\|X_0\|^4 + 4c\|Y_0\|^2$, and $\sigma = -(p/2 + q)$.

Let us define

$$Z_1 = Z_1(v, v_0) = 2c\left[\|X_0\|^2 + \|X\|^2 + 2B(X_0, \theta X)\right]$$

and

$$Z_2 = Z_2(v, v_0) = c^2\left[\|X\|^4 + 2\|X_0\|^2\|X\|^2 + 4B^2(X_0, \theta X)\right] + 4\|X_0\|^2B(X_0, \theta X) + 4\|X\|^2B(X_0, \theta X)$$

$$+ 4c\left[\|Y\|^2 + \|X_0\|^2 + 2B(Y_0, \theta X)\right] + 2B(Y_0, \theta \{X, X_0\}) + 2B(Y, \theta \{X, X_0\}).$$
We may now write
\[ P(v_o, K, v_0) = \left( e^{2t|v_0|} \right)^{s} (1 + Z)^{s}, \]
where
\[ Z = \frac{1 + e^{2rZ_1} + e^{4rZ_2}}{e^{4t|v_0|}}, \]
is a function on \( G/K \times V \). Finally, we call \( \tilde{Z} \) the expression that we obtain in (2) if we take the absolute value of each term making up \( Z_1 \) and \( Z_2 \).

Let \( v_o, K \) be a fixed element in \( G/K \). Since \( \tilde{Z} \) decreases to zero when \( |v_0| \) tends to infinity, it is possible to choose a least positive constant \( C_1 \) (depending on \( v_o, K \)) in such a way that, for every \( v_0 \in V \) with \( |v_0| > C_1 \), then \( \tilde{Z} < \frac{1}{2} \). Obviously, for the same \( v_0 \)'s, also the absolute value of \( Z \) is less than or equal to \( \frac{1}{2} \). Thus, for every \( v_o, K \) fixed, the following identity
\[ P(v_o, K, v_0) = \left( e^{2t|v_0|} \right)^{s} \sum_{j \geq 0} (-1)^{j}b_jZ^j, \]
where
\[ b_j = \frac{(-\sigma + j - 1)!}{(-\sigma - 1)!j!}, \]
is satisfied for all \( v_0 \) such that \( |v_0| > C_1 \).

Remarks. (1) Let \( v \in V \) be a fixed element and let \( t \) belong to the interval \([0, \infty)\). Since the constant \( C_1 \), depending on \( v_o, K \), decreases when \( t \) goes to infinity, there exists a least positive constant \( C_2 \), depending only on \( v \), such that if \( v_0 \in V \) and \( |v_0| > C_2 \), then \( \tilde{Z} < \frac{1}{2} \) for every positive \( t \).

(2) If \( H \) is a compact set in \( G/K \), it is possible to choose a positive constant \( C_3 \), depending on \( H \), in such a way that for all \( v_0 \in V \) with \( |v_0| > C_3 \), then \( \tilde{Z} < \frac{1}{2} \) for every \( v_o, K \in H \).

(3) Obviously the series \( \sum_{j \geq 0} (-1)^j b_j Z^j \) converges where the series \( \sum_{j \geq 0} b_j \tilde{Z}^j \) converges. Moreover, the series \( \sum_{j \geq 0} b_j 2^{-j} \) is convergent.

In order to reorder the series expression for the Poisson kernel \( P \), we consider the action of \( A \) on \( V \) defined by
\[ a \cdot v_0 = \exp(e^sX_0)\exp(e^{2s}Y_0), \]
where \( v_0 = \exp X_0 \exp Y_0 \). We assume also \( s > 0 \). The expression of \( P \) as a function on \( G/K \times (A \cdot V) \) is then the following:
\[ P(v_o, K, a \cdot v_0) \]
\[ = \left[ \frac{e^{2t}}{(1 + ce^{2s}\|e^sX_0 - X\|^2 + 4ce^{4s}\|e^{2s}Y_0 - Y + e^s\{X, X_0\}\|^2} \right]^{-s}. \]
Let us define
\[ R_1 = R_1(v, a \cdot v_0) = 2c \left[ e^{2s}\|X_0\|^2 + \|X\|^2 + 2e^sB(X_0, \theta X) \right]. \]
and

\[ R_2 = R_2(v, a_s \cdot v_0) = c^2 [\|X\|^4 + 2e^{2s}\|X_0\|^2\|X\|^2 + 4e^{2s}B(X_0, \theta X) \\
+ 4e^{3s}\|X_0\|^2B(X_0, \theta X) + 4e^s\|X\|^2B(X_0, \theta X) \]
\[ + 4e^s\|Y\|^2 + e^{2s}\|\{X, X_0\}\|^2 + 2e^{2s}B(Y_0, \theta Y) \\
- 2e^{3s}B(Y_0, \theta \{X, X_0\}) + 2e^sB(Y, \theta \{X, X_0\})]. \]

So we can express

\[ P(va, K, a_s \cdot v_0) = \left( e^{2(\ell + 2s)}|v_0| \right)^{\sigma} \left[ 1 + R \right]^\sigma \\
= \left( e^{2(\ell + 2s)}|v_0| \right)^{\sigma} \sum_{j \geq 0} (-1)^j b_j R^j, \]

where

\[ R = \frac{1 + e^{2s}R_1 + e^{4s}R_2}{e^{(\ell + 2s)}|v_0|}, \]

is a function on \( G/K \times (A \cdot V) \). Collecting together the elements of the sum with the same homogeneity in \( e^s \), we get

\[ P(va, K, a_s \cdot v_0) = \left( e^{2(\ell + 2s)}|v_0| \right)^{\sigma} \sum_{j \geq 0} h_j(va, K, v_0)e^{-js} \\
= \left( e^{2s}|v_0| \right)^{\sigma} \sum_{j \geq 0} h_j(va, K, v_0)e^{(4\sigma - j)s}, \]

where the functions \( h_j \) are defined in such a way that the identity is true when the series is convergent. We define also

\[ H_j(va, K, v_0) = \left( e^{2s}|v_0| \right)^{\sigma} h_j(va, K, v_0). \]

If, in particular, \( s = 0 \) we have

\[ P(va, K, v_0) = \sum_{j \geq 0} H_j(va, K, v_0). \]

3. The \( W_f \) kernel. Let \( H_j : G/K \times V \to \mathbb{R} \) be the functions defined in §2. They are piecewise continuous and bounded on \( V \). Now we want to prove that they are also harmonic as functions on \( G/K \). Let \( v_0 \) be a fixed element in \( V \). Then, for every \( va, K \) in \( G/K \) we choose a positive real number \( s \) large enough so that the following identity holds:

\[ P(va, K, a_s \cdot v_0) = \sum_{j \geq 0} H_j(va, K, v_0)e^{(4\sigma - j)s}. \]

Taking the limit on both sides as \( s \) tends to infinity we find

\[ H_0(va, K, v_0) = \lim_{s \to \infty} P(va, K, a_s \cdot v_0)e^{4\sigma}. \]

Since for every fixed \( s \) the function \( P \) is harmonic, \( H_0(\cdot, v_0) \) is also harmonic on \( G/K \). In the same way, we obtain, for \( j = 1, 2, \ldots, \)
\[ H_j(va,K, v_0) = \lim_{s \to \infty} e^{is} \left[ P(va,K, a_s \cdot v_0)e^{4es} - \sum_{r=0}^{j-1} H_r(va,K, v_0)e^{-rs} \right]. \]

The harmonicity of the function \( H_j(\cdot, v_0) \) on \( G/K \) is thus proved.

Let us consider now the function \( f \) we want to extend. Let \( \varphi \) be a positive continuous function from \( V \) into \( \mathbb{R} \) such that

\[ |f(v_0)|\varphi(v_0) < |v_0|^{-2+\epsilon/2}, \]

for every \( v_0 \in V \). Let us define a nonnegative integer valued function \( J \) on \( V \) by \( J(v_0) = J_0 \), where \( J_0 \) is the least integer for which the following inequality

\[ \sum_{j > (J_0 - 1)/2} b_j 2^{-j} < \varphi(v_1), \]

holds for every \( v_1 \in V \) such that \( 1 < |v_1| < |v_0| \) and where \( J_0 = 0 \) if \( |v_0| < 1 \).

The function \( J \) is well defined and depends on the choice of \( \varphi \). Moreover, if \( v_1, v_2 \in V \) and \( |v_1| < |v_2| \), then \( J(v_1) < J(v_2) \).

We define now a function \( Q \) on \( G/K \times V \) by

\[ Q(va,K, v_0) = \sum_{j < J_0} H_j(va,K, v_0). \]

Finally, we define on \( G/K \times V \) the kernel \( W_f \) associated to the function \( f \) by

\[ W_f(va,K, v_0) = P(va,K, v_0) - Q(va,K, v_0). \]

We estimate this kernel for a fixed element \( va,K \) in \( G/K \). Let us suppose \( C_1 > 1 \). For any \( v_0 \in V \) such that \( |v_0| > C_1 \) we have

\[ |W_f(va,K, v_0)| \leq \left( e^{2|v_0|} \right)^\sigma \sum_{j > \frac{1}{2}(J_0 - 1)} b_j 2^{-j} < \left( e^{2|v_0|} \right)^\sigma \varphi(v_0). \]

**Remarks.** (1) If \( v \in V \) is fixed and \( a_t \) varies in \( A \) when \( t \) belongs to the interval \( [0, \infty) \) we have that

\[ |W_f(va,K, v_0)| < \left( e^{2|v_0|} \right)^\sigma \varphi(v_0), \]

for all \( v_0 \in V \) such that \( |v_0| > C_2 \).

(2) Analogously, when \( va,K \) varies in a compact set \( H \) of \( G/K \), there exists a positive constant \( D \) such that, for every \( v_0 \in V \) with \( |v_0| > C_3 \), we have

\[ |W_f(va,K, v_0)| < D\varphi(v_0). \]

**4. Proof of the theorem.** Let \( T \in \mathbb{R}^+ \). We define \( E(T) = \{ v_0 \in V : |v_0| < T \} \).

(i) Let \( va,K \in G/K \) be a fixed element and suppose \( T > \max(1, C_1) \). We can write
The first integral is finite because \( Pf \) is a bounded continuous function on the compact set \( E(T) \). The second integral is also finite because on \( E(T) \), \( Qf \) is a finite sum of bounded and measurable functions. For the third integral we have

\[
\int_{V \setminus E(T)} |W_f(\nu a, K, v_0)|f(v_0)| dv_0 < \int_{V \setminus E(T)} \left( e^{2t|v_0|} \right)^{\alpha} |\varphi(v_0)|f(v_0)| dv_0 < (e^{2t})^\alpha \int_{V \setminus E(T)} |v_0|^{-2+\alpha/2} dv_0 = \frac{e^{2t\alpha}}{T},
\]

which is finite. So (i) is proved.

(ii) Fix \( \nu \in \mathcal{V} \). Take \( T > \max(1, C_2) \) and observe that \( T > |\nu| \). We know that

\[
\lim_{t \to \infty} \int_{\mathcal{V}} P(\nu a, K, v_0)f(v_0)\chi_{E(T)}(v_0) dv_0 = f(\nu),
\]

if \( \chi_{E(T)} \) is the characteristic function of the compact set \( E(T) \). We can estimate

\[
\lim_{t \to \infty} \left| \int_{E(T)} Q(\nu a, K, v_0)f(v_0) dv_0 \right| < \lim_{t \to \infty} \left( e^{2t|v_0|} \right)^{\alpha} \sum_{j \leq J_0} |h_j(\nu a, K, v_0)f(v_0)| dv_0 = 0,
\]

because \( |h_jf| \) are bounded and measurable functions on \( E(T) \) and the sum is finite on \( E(T) \).

Finally, we estimate

\[
\lim_{t \to \infty} \int_{V \setminus E(T)} |W_f(\nu a, K, v_0)|f(v_0)| dv_0 < \frac{1}{T} \lim_{t \to \infty} e^{2t\alpha} = 0.
\]

So we have proved that \( \lim_{t \to \infty} F(\nu a, K) = f(\nu) \), for every \( \nu a, K \in G/K \).

(iii) Let \( H \) be a fixed compact set in \( G/K \). Let \( T \) be any integer such that \( T > \max(1, C_3) \). We have

\[
\lim_{T \to \infty} \int_{V \setminus E(T)} |W_f(\nu a, K, v_0)|f(v_0)| dv_0 < D \lim_{T \to \infty} \frac{1}{T} = 0
\]

uniformly on \( H \). So, uniformly on \( H \), we have

\[
F(\nu a, K) = \lim_{T \to \infty} \int_{E(T)} P(\nu a, K, v_0)f(v_0) dv_0
\]

\[
- \lim_{T \to \infty} \int_{E(T)} Q(\nu a, K, v_0)f(v_0) dv_0.
\]
Both functions in (3) are harmonic on $G/K$, so (iii) is proved.

Remark. The reader may wish to compare our "explicit" construction with the more general techniques of [4].

REFERENCES


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