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THE VERY WELL POISED ${}_6\psi_6$

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ABSTRACT. A simple proof is given for Bailey's sum of the very well poised ${}_6\psi_6$ and then this is shown to contain a q-extension of the partial fraction decomposition of $\csc \pi x$.

1. Introduction. Bailey's sum of the very well poised ${}_6\psi_6$ is

$$\sum_{-\infty}^{\infty} \frac{(1-aq^{2n})}{(1-a)} \frac{(b;q)_n(c;q)_n(d;q)_n(e;q)_n}{(aq/b;q)_n(aq/c;q)_n(aq/d;q)_n(aq/e;q)_n}\left(\frac{a^2q}{bcde}\right)^n$$
$$= \frac{(aq;q)_\infty(aq/bc;q)_\infty(aq/bd;q)_\infty(aq/be;q)_\infty(aq/cd;q)_\infty(aq/ce;q)_\infty(aq/de;q)_\infty(q;q)_\infty(q/a;q)_\infty}{(q/b;q)_\infty(q/c;q)_\infty(q/d;q)_\infty(q/e;q)_\infty(aq/b;q)_\infty(aq/c;q)_\infty(aq/d;q)_\infty(aq/e;q)_\infty(a^2q/bcde;q)_\infty} \tag{1.1}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1-aq^n), \qquad |q|<1, \tag{1.2}$$

and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad |q|<1. \tag{1.3}$$

Andrews [1, §3] gives a proof of this which is more elementary than the three previous proofs and then gives a sample of the type of results that can be obtained from it. Other q-series identities that follow from the ${}_6\psi_6$ are given by Slater [9]. She remarks that it was only possible to find some of her results after learning of this formula from Bailey. The special case $e = q$, which is the very well poised ${}_6\varphi_5$, was not sufficient for her purposes. It is easy to derive the ${}_6\varphi_5$ summation formula, so it is satisfying, and somewhat surprising, to see that the ${}_6\psi_6$ summation formula can be obtained as a simple consequence of the special case $e = q$. The method we use has been used before to extend another result for a power series to a Laurent series [6]. All that is necessary is to show that two analytic functions agree infinitely often near a point that is an interior point of the set of analyticity. The method is very old and it has not been used on these problems because attention was directed to the wrong variable.

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2. Watson's transformation. Watson [10] found a proof of the Rogers-Ramanujan identities by first extending a transformation of Whipple from hypergeometric series to basic hypergeometric series and then letting all but one of the parameters go to infinity. His transformation has many other applications. It is

$$
{}_8\varphi_7\left(\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix}; q, \frac{a^2q^{n+2}}{bcde}\right)
$$
$$
= \frac{(aq;q)_n(aq/de;q)_n}{(aq/d;q)_n(aq/e;q)_n}\,{}_4\varphi_3\left(\begin{matrix} aq/bc, & d, & e, & q^{-n} \\ de/aq^n, & aq/b, & aq/c \end{matrix}; q, q\right). \tag{2.1}
$$

A simple derivation of (2.1) from orthogonal polynomials is given in [2]. Formula (2.1) can be rewritten as

$$
{}_8\varphi_7\left(\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & f \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right)
$$
$$
= \frac{(aq;q)_\infty(aq/de;q)_\infty(aq/df;q)_\infty(aq/ef;q)_\infty}{(aq/d;q)_\infty(aq/e;q)_\infty(aq/f;q)_\infty(aq/def;q)_\infty}\,{}_4\varphi_3\left(\begin{matrix} aq/bc, & d, & e, & f \\ def/a, & aq/b, & aq/c \end{matrix}; q, q\right). \tag{2.2}
$$

Watson [10] suggested that (2.2) should hold when the left-hand side does not terminate as long as the series on the right-hand side does terminate. Bailey [4] gave a proof of this. The following argument shows that Watson could have concluded this directly from (2.1). Let $1/f = z$. Formula (2.2) can be written as

$$
\sum_{n=0}^{\infty} a_n \frac{(z-1)(z-q)\cdots(z-q^{n-1})}{(1-aqz)\cdots(1-aq^nz)} - F(a,d,e)\frac{(aqz/d;q)_\infty(aqz/e;q)_\infty}{(aqz;q)_\infty(aqz/de;q)_\infty}
$$
$$
\times \sum_{n>0} \frac{(aq/bc;q)_n(d;q)_n(e;q)_n}{(aq/b;q)_n(aq/c;q)_n(q;q)_n} q^n \frac{(1-z^{-1})\cdots(1-q^{n-1}/z)}{(1-de/az)\cdots(1-deq^{n-1}/az)} = 0
$$

where a_n and $F(a, d, e)$ are independent of z. If $aq/bc = q^{-k}$ is fixed, then this function is an analytic function of z for $|z| < \min(|aq|^{-1}, |dea^{-1}q^{k-1}|)$. It vanishes when $z = q^j$, $j = 1, 2, \ldots$, and so vanishes identically. Thus (2.2) holds when the series on the right-hand side terminates. When this series does not terminate there is a second term on the right. See [4]. Jackson's sum of the very well poised ${}_6\varphi_5$ is an easy consequence of (2.2). Let $d = aqc^{-1}$. This gives

$$
{}_6\varphi_5\left(\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & e, & f \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/e, & aq/f \end{matrix}; q, \frac{aq}{bef}\right)
$$
$$
= \frac{(aq;q)_\infty(aq/de;q)_\infty(aq/df;q)_\infty(aq/ef;q)_\infty}{(aq/d;q)_\infty(aq/e;q)_\infty(aq/f;q)_\infty(aq/def;q)_\infty}\,{}_3\varphi_2\left(\begin{matrix} d/b, & e, & f \\ def/a, & aq/b \end{matrix}; q, q\right) \tag{2.3}
$$

where $d/b = q^{-k}$. The ${}_3\varphi_2$ in this equality is balanced (Saalschutzian), i.e., it is

$$
{}_3\varphi_2\left(\begin{matrix} q^{-k}, & a, & b \\ c, & abq^{1-k}c^{-1} \end{matrix}; q, q\right),
$$

and so can be summed by

$$ {}_3\varphi_2\left(\begin{matrix} q^{-k}, & a, & b \\ c, & abq^{1-k}c^{-1} \end{matrix}; q, q\right) = \frac{(c/a;q)_k(c/b;q)_k}{(c;q)_k(c/ab;q)_k} $$

or

$$ {}_3\varphi_2\left(\begin{matrix} a, & b, & c \\ d, & abcqd^{-1} \end{matrix}; q, q\right) = \frac{(d/a;q)_\infty(d/b;q)_\infty(d/c;q)_\infty(d/abc;q)_\infty}{(d;q)_\infty(d/ab;q)_\infty(d/ac;q)_\infty(d/bc;q)_\infty}, \tag{2.4} $$

when this series terminates. Using (2.4) in (2.3) gives

$$ {}_6\varphi_5\left(\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & e, & f \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/e, & aq/f \end{matrix}; q, \frac{aq}{bef}\right) $$
$$ = \frac{(aq;q)_\infty(aq/ef;q)_\infty(aq/be;q)_\infty(aq/bf;q)_\infty}{(aq/b;q)_\infty(aq/e;q)_\infty(aq/f;q)_\infty(aq/bef;q)_\infty}. \tag{2.5} $$

Observe that the parameter d which was used to terminate the right-hand side of (2.3) has disappeared. The only condition needed in (2.5) is $|aq| < |bef|$ to insure convergence and then isolated conditions on b, e and f to avoid poles.

3. Bailey's sum. Bailey [5] found the sum of

$$ \sum_{-\infty}^{\infty} \frac{(q\sqrt{a};q)_n(-q\sqrt{a};q)_n(b;q)_n(c;q)_n(d;q)_n(e;q)_n}{(\sqrt{a};q)_n(-\sqrt{a};q)_n(aq/b;q)_n(aq/c;q)_n(aq/d;q)_n(aq/e;q)_n}\left(\frac{a^2q}{bcde}\right)^n. \tag{3.1} $$

When $e = a$ the series (3.1) becomes the series in (2.5) since $1/(q;q)_n = 0$ when $n = -1, -2, \ldots$. Surprisingly it is possible to obtain the sum of (3.1) directly from (2.5). When $aq/e = q^{m+1}$, or $e = aq^{-m}$, then (3.1) is

$$ \sum_{n=-m}^{\infty} \frac{(qa^{1/2};q)_n(-qa^{1/2};q)_n(b;q)_n(c;q)_n(d;q)_n(aq^{-m};q)_n}{(a^{1/2};q)_n(-a^{1/2};q)_n(aq/b;q)_n(aq/c;q)_n(aq/d;q)_n(q^{m+1};q)_n}\left(\frac{aq^{m+1}}{bcd}\right)^n $$
$$ = \sum_{n=0}^{\infty} \frac{(1-aq^{2n-2m})}{(1-a)} \frac{(b;q)_{n-m}(c:q)_{n-m}(d;q)_{n-m}(aq^{-m};q)_{n-m}}{(aq/b;q)_{n-m}(aq/c;q)_{n-m}(aq/d;q)_{n-m}(a^{m+1};q)_{n-m}}\left(\frac{aq^{m+1}}{bcd}\right)^{n-m} $$
$$ = \frac{(b;q)_{-m}(c;q)_{-m}(d;q)_{-m}(aq^{-m};q)_{-m}(1-aq^{-2m})}{(aq/b;q)_{-m}(aq/c;q)_{-m}(aq/d;q)_{-m}(q^{m+1};q)_{-m}(1-a)}\left(\frac{bcd}{aq^{m+1}}\right)^m $$
$$ \times \sum_{n=0}^{\infty} \frac{(1-aq^{-2m+2n})}{(1-aq^{-2m})} \frac{(bq^{-m};q)_n(cq^{-m};q)_n(dq^{-m};q)_n(aq^{-2m};q)_n}{(aq^{1-m}/b;q)_n(aq^{1-m}/c;q)_n(aq^{1-m}/d;q)_n(q;q)_n}\left(\frac{aq^{m+1}}{bcd}\right)^n. $$

This series is very well poised so it can be summed by (2.5). The resulting infinite products can be simplified, using

$$ (a;q)_{-n} = \frac{(a;q)_\infty}{(aq^{-n};q)_\infty} = \frac{1}{(aq^{-n};q)_n} = \frac{(-1)^n q^{\binom{n+1}{2}}}{(q/a;q)_n a^n}. $$

When this is done the resulting identity is (1.1) with $e = aq^{-m}$. If $x = ae^{-1}$ then both sides of (1.1) are analytic functions of x for

$$ |x| < \min(|aq^{-1}|, |q^{-1}|, |bcda^{-1}q^{-1}|), $$

and they agree when $x = q^m$, $m = 0, 1, \ldots$. Thus they must be identically equal.

4. An identity of Jacobi. The q-gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad 0<q<1. \tag{4.1}$$

F. H. Jackson [7] introduced this function and gave some properties. Others are given in [3]. Most of the classical properties of the gamma function have been extended to the q-gamma function. Here is another extension of

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{4.2}$$

In formula (1.1) let $e \to \infty$, $d = (aq)^{1/2}$. This sum is then

$$\sum_{-\infty}^{\infty} \frac{(1-aq^{2n})}{(1-a)} \frac{(b;q)_n(c;q)_n(-1)^n q^{\binom{n}{2}}}{(aq/b;q)_n(aq/c;q)_n}\left(\frac{a^{3/2}q^{1/2}}{bc}\right)^n$$
$$= \frac{(aq;q)_\infty(aq/bc;q)_\infty((aq)^{1/2}/b;q)_\infty((aq)^{1/2}/c;q)_\infty(q;q)_\infty(q/a;q)_\infty}{(q/b;q)_\infty(q/c;q)_\infty((q/a)^{1/2};q)_\infty(aq/b;q)_\infty(aq/c;q)_\infty((aq)^{1/2};q)_\infty}. \tag{4.3}$$

Next write $(1-q/a) = (1+(qa^{-1})^{1/2})(1-(aq^{-1})^{1/2})$, so that

$$\frac{(q/a;q)_\infty}{((q/a)^{1/2};q)_\infty} = \frac{(q^2a^{-1};q)_\infty}{(q^{3/2}a^{-1/2};q)_\infty}\left(1+(qa^{-1})^{1/2}\right).$$

Then let $a \to q$. The result is

$$\sum_{-\infty}^{\infty} \frac{(1-q^{2n+1})}{(1-q)} \frac{(b;q)_n(c;q)_n(-1)^n}{(q^2/b;q)_n(q^2/c;q)_n} \frac{q^{(n^2+3n)/2}}{(bc)^n} = 2\frac{(q^2;q)_\infty(q^2/bc;q)_\infty}{(q^2/b;q)_\infty(q^2/c;q)_\infty}. \tag{4.4}$$

Replace q by q^2 and let $b = q^{1+2x}$, $c = q^{1-2x}$. Formula (4.4) is then

$$\sum_{-\infty}^{\infty} \frac{(1-q^{4n+2})}{(1-q^2)} \frac{(q^{1+2x};q^2)_n(q^{1-2x};q^2)_n(-1)^n q^{n^2+n}}{(q^{3+2x};q^2)_n(q^{3-2x};q^2)_n}$$
$$= 2\frac{(q^4;q^2)_\infty(q^2;q^2)_\infty}{(q^{3+2x};q^2)_\infty(q^{3-2x};q^2)_\infty} \tag{4.5}$$

or

$$\frac{1}{2}\sum_{-\infty}^{\infty} \frac{(1-q^{4n+2})(-1)^n q^{n^2+n}}{(1-q^{2n+1+2x})(1-q^{2n+1-2x})} = \frac{(q^2;q^2)_\infty^2}{(q^{1+2x};q^2)_\infty(q^{1-2x};q^2)_\infty}. \tag{4.6}$$

Using the q-gamma function this can be rewritten as

$$\Gamma_{q^2}(1/2+x)\Gamma_{q^2}(1/2-x) = \frac{(1-q^2)}{2}\sum_{-\infty}^{\infty} \frac{(1-q^{4n+2})(-1)^n q^{n^2+n}}{(1-q^{2n+1+2x})(1-q^{2n+1-2x})}. \tag{4.7}$$

This can be rewritten by breaking up the terms in the sum by partial fractions and rearranging. The result is

$$\Gamma_{q^2}(1/2+x)\Gamma_{q^2}(1/2-x) = (1-q^2)\sum_{-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1-q^{2n+1+2x})}. \tag{4.8}$$

Replacing $2x+1$ by $2x$ gives

$$\Gamma_q(x)\Gamma_q(1-x) = \sum_{-\infty}^{\infty} \frac{(1-q)(-1)^n q^{n(n+1)/2}}{(1-q^{n+x})}. \tag{4.9}$$

When $q \to 1^-$ this reduces to

$$\Gamma(x)\Gamma(1 - x) = \sum_{-\infty}^{\infty} \frac{(-1)^n}{n + x} = \frac{\pi}{\sin \pi x}. \tag{4.10}$$

A formula equivalent to (4.7) was given by Jacobi [**8**, p. 187, formula (11)]. There are other ways to prove (4.9), but this proof is interesting for it gives one further example of a nice result that follows from the ${}_6\psi_6$ formula. Other formulas that are easy corollaries of this summation formula are the formulas for the number of representations of n as the sum of 2, 4, 6 or 8 squares. See Andrews [**1**].

Bibliography

1. G. Andrews, *Applications of basic hypergeometric series*, SIAM Rev. **16** (1974), 441–484.

2. G. Andrews and R. Askey, *Enumeration of partitions: the role of Eulerian series and q-orthogonal polynomials*, Higher Combinatorics, edited by M. Aigner, Reidel, Dordrecht and Boston, 1977, 3–26.

3. R. Askey, *The q-gamma and q-beta functions*, Applicable Anal. **8** (1978), 125–141.

4. W. N. Bailey, *Generalized hypergeometric series*, Cambridge Univ. Press, Cambridge, 1935.

5. ______, *Series of hypergeometric type which are infinite in both directions*, Quart. J. Math. Oxford Ser. 7 (1936), 105–115.

6. M. E.-H. Ismail, *A simple proof of Ramanujan's ${}_1\psi_1$ sum*, Proc. Amer. Math. Soc. **63** (1977), 185–186.

7. F. H. Jackson, *On q-definite integrals*, Quart. J. Pure and Appl. Math. **41** (1910), 193–203.

8. C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Regiomontis, fratrum Borntragger, 1829.

9. L. J. Slater, *A new proof of Rogers's transformations of infinite series*, Proc. London Math. Soc. (2) **53** (1951), 460–475.

10. G. N. Watson, *A new proof of the Rogers-Ramanujan identities*, J. London Math. Soc. **4** (1929), 4–9.

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