

ABSOLUTE ABEL SUMMABILITY AND CAPACITY

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ABSTRACT. Precise limits on the size of exceptional sets for which functions in the Lebesgue class, \mathcal{L}_α^p , can fail to be absolutely Abel summable are given in terms of Bessel capacity.

1. Introduction. In this note we present two theorems which generalize results of [1] and [2]. We do this in the context of the Lebesgue classes \mathcal{L}_α^p and the capacity theory developed in [3]. Here we introduce those facts from [3] which are for the most part contained in Theorem 16 of that paper.

Let $Q_k = \{x \in E_k: -\frac{1}{2} < x_i < \frac{1}{2}, i = 1, 2, \dots, k\}$ and $k > 2$ be the torus in k -dimensional Euclidean space. Let $0 < \alpha < k$, and g_α be the kernel of the Bessel potential which is given by the positive function whose Fourier coefficients are $\hat{g}_\alpha(n) = (1 + 4\pi^2|n|^2)^{-\alpha/2}$ where n is a point in the k -dimensional integral lattice plane. Let \mathcal{L}_1^+ be the space of all nonnegative Radon measures of finite total variation on Q_k . If $\nu \in \mathcal{L}_1^+$, $\|\nu\|$ denotes its total variation. For $\nu \in \mathcal{L}_1^+$, $g_\alpha(\nu, x) = (g_\alpha * \nu)(x)$. For $Z \subset Q_k$ an analytic set the capacity of Z is defined for $1 < p < \infty$ by

$$c_{\alpha,p}(Z) = \sup \|\nu\|, \tag{1.1}$$

where the supremum is taken over all those $\nu \in \mathcal{L}_1^+$ concentrated on Z for which $\|g_\alpha(\nu, \cdot)\|_p < 1$ and $p' = p/(p - 1)$.

If Z has positive capacity then there is a nontrivial $\mu \in \mathcal{L}_1^+$ satisfying the variational problem of (1.1), the function f defined by

$$f^{p-1}(x) = (c_{\alpha,p}(Z))^{p-1} g_\alpha(\mu, x) \tag{1.2}$$

is in L^p , and $\|f\|_p = c_{\alpha,p}(Z)$. Moreover μ is concentrated on the set $Z \cap \{x: (f * g_\alpha)(x) = 1\} = Z_0$ and the set $Z - Z_0$ has zero $c_{\alpha,p}$ capacity. Such f and μ are called capacity distributions for Z . Let $f_0 = f/\|f\|_p$ and $\mu_0 = \mu/\|\mu\|$. Finally, for any function h satisfying $(h * g_\alpha)(x) \geq 1$ on Z and $\|h\|_p = 1$, and any measure ν concentrated on Z satisfying $\|\nu\| = 1$, and $\|g_\alpha(\nu, \cdot)\|_{p'} < 1$ we have

$$g_\alpha(\mu_0, h) \leq g_\alpha(\mu_0, f_0) \leq g_\alpha(\nu, f_0) \tag{1.3}$$

where $g_\alpha(\mu, f) = \iint f(y)g_\alpha(x - y) d\mu(x) dy$.

We say a function f belongs to the Lebesgue class \mathcal{L}_α^p if f can be written as $f_0 * g_\alpha$ for some f_0 in $L^p = L^p(Q_k)$.

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2. Absolute Abel summability. In this section we deal with absolute Abel summability of multiple Fourier series. We say the Fourier series of a function f is absolutely Abel summable at a point x^0 if

$$\int_0^1 \left| \frac{\partial f}{\partial t}(x^0, t) \right| dt < \infty$$

where

$$f(x, t) = \sum_m \hat{f}(m) e^{2\pi(im \cdot x - |m|t)} \quad \text{for } t > 0.$$

The theorems we prove are the following.

THEOREM 1. *Let f be a function of class $\mathcal{L}_\alpha^p(Q_k)$. Then f is absolutely Abel summable except possibly on a set of zero $c_{\alpha,p}$ capacity.*

THEOREM 2. *Let Z be a closed set in Q_k which is of zero $c_{\alpha,p}$ capacity. Then there exists a function f in $\mathcal{L}_\alpha^p(Q_k)$ such that*

$$\int_0^1 \left| \frac{\partial f(x, t)}{\partial t} \right| dt = \infty \quad \text{for each } x \in Z.$$

It should be noted that in [2] these theorems are given for the case $p = 2$ and ordinary capacity, which is motivated by one dimensional results of [1].

Let

$$g_\alpha(x, t) = \sum_m \frac{e^{2\pi im \cdot x - 2\pi |m|t}}{(1 + 4\pi^2 |m|^2)^{\alpha/2}}, \quad t > 0.$$

We have the following which can be obtained from an application of the Poisson summation formula and the properties of Bessel potentials listed in §7 of [3].

(2.1) For $\alpha > 0$, $g_\alpha(x, t) \rightarrow g_\alpha(x)$ provided x is not a lattice point, as $t \rightarrow 0$.

(2.2) The function $g_\alpha(x)$ is continuous if x is not a lattice point, $g_\alpha \in L^1$ and $g_\alpha > 0$.

$$\int_0^1 \left| \frac{\partial g_\alpha}{\partial t}(x, t) \right| dt < g_\alpha(x) + c(k) \quad \text{where } c(k) = \sum_m e^{-2\pi |m|}. \quad (2.3)$$

To see that (2.3) holds, note that

$$\int_0^1 \left| \frac{\partial g_\alpha(x, t)}{\partial t} \right| dt < \lim_{\epsilon \rightarrow 0} \int_\epsilon^1 \left| \frac{\partial g_\alpha(x, t)}{\partial t} \right| dt$$

and begin by applying the Poisson summation formula to justify that

$$\frac{\partial g_\alpha(x, t)}{\partial t} = \int_{y \in \mathbb{R}^k} g_\alpha(x - y) \frac{\partial P_t}{\partial t}(y) dy$$

$$\text{where } P_t(y) = C \frac{t}{(t^2 + |y|^2)^{(k+1)/2}}.$$

The positivity of g_α is used several times in the following. It follows that

$$\begin{aligned} \int_\epsilon^1 \left| \frac{\partial g_\alpha(x, t)}{\partial t} \right| dt &< \int_\epsilon^1 \int g_\alpha(x - y) \left| \frac{\partial P_t}{\partial t}(y) \right| dy dt \\ &< \left\{ \int_\epsilon^1 \int_{|y| < 2\sqrt{k}} + \int_\epsilon^1 \int_{|y| > 2\sqrt{k}} \right\} g_\alpha(x - y) \left| \frac{\partial P_t}{\partial t} \right| dy dt \\ &\equiv A + B. \end{aligned}$$

Direct calculation of $\partial P_t / \partial t$ shows that $\partial P_t(y) / \partial t > 0$ if and only if $|y| > \sqrt{k} t$. Define

$$l = l(y, \epsilon) = \begin{cases} \epsilon, & |y| < \epsilon\sqrt{k}, \\ 1, & |y| > \sqrt{k}, \\ |y|/\sqrt{k}, & \epsilon\sqrt{k} < |y| < \sqrt{k}. \end{cases}$$

Then it follows that

$$\begin{aligned} A &= - \int_{|y| < 2\sqrt{k}} g_\alpha(x - y) \int_\epsilon^l \frac{\partial P_t}{\partial t}(y) dt dy \\ &\quad + \int_{|y| < 2\sqrt{k}} g_\alpha(x - y) \int_l^1 \frac{\partial P_t}{\partial t}(y) dt dy \\ &= \int_{|y| < 2\sqrt{k}} g_\alpha(x - y) [P_\epsilon(y) - P_l(y)] dy \\ &\quad + \int_{|y| < 2\sqrt{k}} g_\alpha(x - y) [P_l(y) - P_1(y)] dy \\ &< \int_{|y| < 2\sqrt{k}} g_\alpha(x - y) [P_\epsilon(y) + P_l(y)] dy \\ &< g_\alpha(x, \epsilon) + \int_{|y| < 2\sqrt{k}} g_\alpha(x - y) P_l(y) dy. \end{aligned}$$

Similar considerations for $\partial P_t(y) / \partial t$ lead to

$$B = \int_{|y| > 2\sqrt{k}} g_\alpha(x - y) [P_l(y) - P_\epsilon(y)] dy < \int_{|y| > 2\sqrt{k}} g_\alpha(x - y) P_l(y) dy.$$

Combining the estimates for A and B gives

$$\int_0^1 \left| \frac{\partial g_\alpha(x, t)}{\partial t} \right| dt < \lim_{\epsilon \rightarrow 0} g_\alpha(x, \epsilon) + g_\alpha(x, 1).$$

Again by the Poisson summation formula

$$g_\alpha(x, 1) < \left| \sum_m \frac{1}{(1 + 4\pi^2|m|^2)^{\alpha/2}} e^{-2\pi|m|} e^{imx} \right| < c(k).$$

Next, let

$$\begin{aligned} f(x, t) &= \sum_m \hat{f}(m) e^{2\pi i m \cdot x - 2\pi |m| t} \\ &= \sum_m \hat{f}_0(m) \frac{1}{(1 + 4\pi^2 |m|^2)^{\alpha/2}} e^{2\pi i m \cdot x - 2\pi |m| t} \\ &= \int_{Q_k} f_0(y) \cdot g_\alpha(x - y, t) dy. \end{aligned}$$

It follows that

$$\frac{\partial f}{\partial t}(x, t) = \int_{Q_k} f_0(y) \frac{\partial g_\alpha}{\partial t}(x - y, t) dy \quad (2.4)$$

so that

$$\begin{aligned} \int_0^1 \left| \frac{\partial f}{\partial t}(x, t) \right| dt &< \int_{Q_k} |f_0(y)| \int_0^1 \left| \frac{\partial g_\alpha}{\partial t}(x - y, t) \right| dy dt \\ &< c(k) \int_{Q_k} |f_0(y)| dy + (|f_0| * g_\alpha)(x). \end{aligned} \quad (2.5)$$

We now present the proof of Theorem 1.

PROOF OF THEOREM 1. Let $E_\infty = \{x: \int_0^1 |\partial f(x, t)/\partial t| dt = \infty\}$ and suppose that E_∞ has positive $c_{\alpha,p}$ capacity. Then there exists a measure μ (capacitary distribution) concentrated on $B_\infty = \{x: g_\alpha(x, f_\mu) = 1\} \cap E_\infty$ where $f_\mu(y) = c_{\alpha,p}(E_\infty) [g_\alpha(\mu, y)]^{1/(p-1)}$. We use the notation compatible with §1:

$$\begin{aligned} g_\alpha(\mu, y) &= \int_{Q_k} g_\alpha(x - y) d\mu(x), \\ g_\alpha(x, f_\mu) &= \int_{Q_k} g_\alpha(x - y) f_\mu(y) dy, \\ c_{\alpha,p}(E_\infty) &= \int f_\mu^p(y) dy = c_{\alpha,p}(E_\infty) \int g_\alpha(\mu, y)^{p'} dy, \end{aligned}$$

and $\|\mu\| = c_{\alpha,p}(E_\infty) > 0$.

In this case, let $\nu = c_{\alpha,p}^{-1}(E_\infty)\mu$ so that $\|\nu\| = 1$ and $\nu(Q_k - E_\infty) = 0$. Then

$$\begin{aligned} \iint |f_0(y)| g_\alpha(x - y) d\nu(x) dy &= \int |f_0(y)| \cdot g_\alpha(\nu, y) dy \\ &< \|f_0\|_p \cdot \|g_\alpha(\nu, \cdot)\|_{p'} = \|f_0\|_p \cdot c_{\alpha,p}^{-1}(E_\infty). \end{aligned} \quad (2.6)$$

The last inequality follows from the fact that $\|g_\alpha(\mu, \cdot)\|_{p'} < 1$.

We now have that

$$\int_{Q_k} \int_0^1 \left| \frac{\partial f}{\partial t}(x, t) \right| dt d\nu(x) = \infty, \quad (2.7)$$

while on the other hand, by (2.5) and (2.6), we have

$$\begin{aligned} & \int_{Q_k} \int_0^1 \left| \frac{\partial f}{\partial t}(x, t) \right| dt \, d\nu(x) \\ & < c(k) |Q_k|^{1/p'} \|f_0\|_p + \int |f_0|(y) g_\alpha(x - y) \, d\nu(x) \, dy \\ & = [c(k) |Q_k|^{1/p'} + c_{\alpha,p}^{-1}(E_\infty)] \|f_0\|_p < \infty. \end{aligned} \tag{2.8}$$

This contradiction completes the proof of Theorem 1.

3. Counterexample. We now give the construction of the counterexample for Theorem 2.

Let Z be a closed set in Q_k such that $c_{\alpha,p}(Z) = 0$. Since $c_{\alpha,p}$ is a capacity, we can approximate $c_{\alpha,p}(Z)$ by $c_{\alpha,p}(K_\rho)$ where $K_\rho = \{x: \text{dist}(x, Z) < \rho\}$; that is,

$$c_{\alpha,p}(Z) = \lim_{\rho \rightarrow 0} c_{\alpha,p}(K_\rho). \tag{3.1}$$

For each $\rho > 0$ let μ_ρ and f_ρ be the capacity distributions for K_ρ with $f_\rho(y)^{p-1} = c_{\alpha,p}(K_\rho)^{p-1} g_\alpha(\mu_\rho, y)$ almost everywhere and $f_\rho * g_\alpha(x) = 1$ almost everywhere with respect to capacity $c_{\alpha,p}$ on K_ρ . Then it follows that

$$\int g_\alpha(x - y) \left\{ \int g_\alpha(y - z) \, d\mu_\rho(z) \right\}^{1/(p-1)} dy = c_{\alpha,p}(K_\rho)^{-1} \tag{3.2}$$

almost everywhere in K_ρ with respect to $c_{\alpha,p}$ capacity. We choose a sequence $\{\mu_{\rho_j}\}$ associated in the above fashion such that $c_{\alpha,p}(K_{\rho_j}) = \|\mu_{\rho_j}\| \rightarrow 0$.

Then $\|f_\rho\|_p = c_{\alpha,p}(K_\rho)$. Note that f_ρ is not constant on K_ρ but it has a potential which is equal to one almost everywhere with respect to $c_{\alpha,p}$ capacity. Let

$$f_0^k = \sum_{j=1}^k c_j f_{\rho_j} \quad \text{and} \quad f^k = \sum_{j=1}^k g_\alpha * (c_j f_{\rho_j}). \tag{3.3}$$

Hence f^k is in $\mathcal{L}_\alpha^p(Q_k)$. For c_j use $j^{-\delta} c_{\alpha,p}(K_{\rho_j})^{-1}$ with $\delta > 1$ so that we have $\|f_0^k\|_p < \sum_{j=1}^k j^{-\delta} < \infty$ and $\{f_0^k\}$ forms a Cauchy sequence in $L^p(Q_k)$. Let f_0 denote the limit of f_0^k as k tends to infinity. Since f_{ρ_j} is a positive function, f_0 is also the pointwise limit of f_0^k . Furthermore, for almost every x in K_{ρ_k} with respect to $c_{\alpha,p}$ capacity, we have

$$(f_0^k * g_\alpha)(x) > \sum_{j=1}^k j^{-\delta} c_{\alpha,p}(K_{\rho_j})^{-1} (f_{\rho_j} * g_\alpha)(x) = \sum_{j=1}^k j^{-\delta} c_{\alpha,p}(K_{\rho_j})^{-1}. \tag{3.4}$$

We choose the sequence $\{\rho_j\}$ such that $c_{\alpha,p}(K_{\rho_j}) < j^{-\delta}$. Then in K_{ρ_k} , $(P_t * f_0^k * g_\alpha)$ tends to $\sum_{j=1}^k j^{-\delta} c_{\alpha,p}(K_{\rho_j})^{-1} > k$ as t tends to zero, since $f_0^k * g_\alpha$ is essentially constant in K_{ρ_k} .

Since

$$|f(x, 1) - f(x, t)| < \int_t^1 \left| \frac{\partial f}{\partial s}(x, s) \right| ds \quad \text{and} \quad |f(x, 1)| < c(k) |Q_k|^{1/p'},$$

it will follow that

$$\int_0^1 \left| \frac{\partial f}{\partial t}(x, t) \right| dt = \infty \quad \text{for } x \text{ in } Z \quad (3.5)$$

if we show

$$\lim_{t \rightarrow 0} f(x, t) = \infty \quad \text{for } x \text{ in } Z. \quad (3.6)$$

But for x in Z we have

$$\lim_{t \rightarrow \infty} \inf f(x, t) > \lim_{t \rightarrow \infty} \inf (P_t * f_0^k * g_\alpha) > k$$

for all k . This concludes the construction of Theorem 2.

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