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ABSOLUTE ABEL SUMMABILITY AND CAPACITY

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ABSTRACT. Precise limits on the size of exceptional sets for which functions in the Lebesgue class, $\mathcal{L}_\alpha^p$, can fail to be absolutely Abel summable are given in terms of Bessel capacity.

1. Introduction. In this note we present two theorems which generalize results of [1] and [2]. We do this in the context of the Lebesgue classes $\mathcal{L}_\alpha^p$ and the capacity theory developed in [3]. Here we introduce those facts from [3] which are for the most part contained in Theorem 16 of that paper.

Let $Q_k = \{x \in E_k: -\frac{1}{2} \leq x_i < \frac{1}{2}, i = 1, 2, \ldots, k\}$ and $k \geq 2$ be the torus in k-dimensional Euclidean space. Let $0 < \alpha < k$, and g_α be the kernel of the Bessel potential which is given by the positive function whose Fourier coefficients are $\hat{g}_\alpha(n) = (1 + 4\pi^2|n|^2)^{-\alpha/2}$ where n is a point in the k-dimensional integral lattice plane. Let $\mathcal{L}_1^+$ be the space of all nonnegative Radon measures of finite total variation on Q_k. If $\nu \in \mathcal{L}_1^+$, $\|\nu\|$ denotes its total variation. For $\nu \in \mathcal{L}_1^+$, $g_\alpha(\nu, x) = (g_\alpha * \nu)(x)$. For $Z \subset Q_k$ an analytic set the capacity of Z is defined for $1 < p < \infty$ by

$$c_{\alpha,p}(Z) = \sup\|\nu\|, \tag{1.1}$$

where the supremum is taken over all those $\nu \in \mathcal{L}_1^+$ concentrated on Z for which $\|g_\alpha(\nu, \cdot)\|_{p'} \leq 1$ and $p' = p/(p-1)$.

If Z has positive capacity then there is a nontrivial $\mu \in \mathcal{L}_1^+$ satisfying the variational problem of (1.1), the function f defined by

$$f^{p-1}(x) = \left(c_{\alpha,p}(Z)\right)^{p-1} g_\alpha(\mu, x) \tag{1.2}$$

is in L^p, and $\|f\|_p = c_{\alpha,p}(Z)$. Moreover μ is concentrated on the set $Z \cap \{x: (f * g_\alpha)(x) = 1\} = Z_0$ and the set $Z - Z_0$ has zero $c_{\alpha,p}$ capacity. Such f and μ are called capacitary distributions for Z. Let $f_0 = f/\|f\|_p$ and $\mu_0 = \mu/\|\mu\|$. Finally, for any function h satisfying $(h * g_\alpha)(x) \geq 1$ on Z and $\|h\|_p = 1$, and any measure ν concentrated on Z satisfying $\|\nu\| = 1$, and $\|g_\alpha(\nu, \cdot)\|_{p'} \leq 1$ we have

$$g_\alpha(\mu_0, h) \geq g_\alpha(\mu_0, f_0) \geq g_\alpha(\nu, f_0) \tag{1.3}$$

where $g_\alpha(\mu, f) = \int\int f(y) g_\alpha(x - y)\, d\mu(x)\, dy$.

We say a function f belongs to the Lebesgue class $\mathcal{L}_\alpha^p$ if f can be written as $f_0 * g_\alpha$ for some f_0 in $L^p = L^p(Q_k)$.

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2. Absolute Abel summability. In this section we deal with absolute Abel summability of multiple Fourier series. We say the Fourier series of a function f is absolutely Abel summable at a point x^0 if

$$\int_0^1 \left| \frac{\partial f}{\partial t}(x^0, t) \right| dt < \infty$$

where

$$f(x, t) = \sum_m \hat{f}(m) e^{2\pi(im\cdot x - |m|t)} \quad \text{for } t > 0.$$

The theorems we prove are the following.

THEOREM 1. *Let f be a function of class $\mathfrak{L}_\alpha^p(Q_k)$. Then f is absolutely Abel summable except possibly on a set of zero $c_{\alpha,p}$ capacity.*

THEOREM 2. *Let Z be a closed set in Q_k which is of zero $c_{\alpha,p}$ capacity. Then there exists a function f in $\mathfrak{L}_\alpha^p(Q_k)$ such that*

$$\int_0^1 \left| \frac{\partial f(x, t)}{\partial t} \right| dt = \infty \quad \textit{for each } x \in Z.$$

It should be noted that in [2] these theorems are given for the case $p = 2$ and ordinary capacity, which is motivated by one dimensional results of [1].

Let

$$g_\alpha(x, t) = \sum_m \frac{e^{2\pi im\cdot x - 2\pi t|m|}}{(1 + 4\pi^2|m|^2)^{\alpha/2}}, \qquad t > 0.$$

We have the following which can be obtained from an application of the Poisson summation formula and the properties of Bessel potentials listed in §7 of [3].

(2.1) For $\alpha > 0$, $g_\alpha(x, t) \to g_\alpha(x)$ provided x is not a lattice point, as $t \to 0$.

(2.2) The function $g_\alpha(x)$ is continuous if x is not a lattice point, $g_\alpha \in L^1$ and $g_\alpha > 0$.

$$\int_0^1 \left| \frac{\partial g_\alpha}{\partial t}(x, t) \right| dt < g_\alpha(x) + c(k) \quad \text{where } c(k) = \sum_m e^{-2\pi|m|}. \tag{2.3}$$

To see that (2.3) holds, note that

$$\int_0^1 \left| \frac{\partial g_\alpha(x, t)}{\partial t} \right| dt < \lim_{\varepsilon \to 0} \int_\varepsilon^1 \left| \frac{\partial g_\alpha(x, t)}{\partial t} \right| dt$$

and begin by applying the Poisson summation formula to justify that

$$\frac{\partial g_\alpha(x, t)}{\partial t} = \int_{y \in \mathbf{R}^k} g_\alpha(x - y) \frac{\partial P_t}{\partial t}(y)\, dy$$

$$\text{where } P_t(y) = C \frac{t}{(t^2 + |y|^2)^{(k+1)/2}}.$$

The positivity of g_α is used several times in the following. It follows that

$$\begin{aligned}\int_\varepsilon^1 \left|\frac{\partial g_\alpha(x,t)}{\partial t}\right| dt &< \int_\varepsilon^1 \int g_\alpha(x-y)\left|\frac{\partial P_t}{\partial t}(y)\right| dy\, dt \\ &< \left\{\int_\varepsilon^1 \int_{|y|<2\sqrt{k}} + \int_\varepsilon^1 \int_{|y|>2\sqrt{k}}\right\} g_\alpha(x-y)\left|\frac{\partial P_t}{\partial t}\right| dy\, dt \\ &\equiv A + B.\end{aligned}$$

Direct calculation of $\partial P_t/\partial t$ shows that $\partial P_t(y)/\partial t > 0$ if and only if $|y| > \sqrt{k}\, t$. Define

$$l = l(y,\varepsilon) = \begin{cases} \varepsilon, & |y| < \varepsilon\sqrt{k}, \\ 1, & |y| > \sqrt{k}, \\ |y|/\sqrt{k}, & \varepsilon\sqrt{k} < |y| < \sqrt{k}. \end{cases}$$

Then it follows that

$$\begin{aligned} A &= -\int_{|y|<2\sqrt{k}} g_\alpha(x-y)\int_\varepsilon^l \frac{\partial P_t}{\partial t}(y)\, dt\, dy \\ &\quad + \int_{|y|<2\sqrt{k}} g_\alpha(x-y)\int_l^1 \frac{\partial P}{\partial t}(y)\, dt\, dy \\ &= \int_{|y|<2\sqrt{k}} g_\alpha(x-y)[P_\varepsilon(y) - P_l(y)]\, dy \\ &\quad + \int_{|y|<2\sqrt{k}} g_\alpha(x-y)[P_1(y) - P_l(y)]\, dy \\ &< \int_{|y|<2\sqrt{k}} g_\alpha(x-y)[P_\varepsilon(y) + P_1(y)]\, dy \\ &< g_\alpha(x,\varepsilon) + \int_{|y|<2\sqrt{k}} g_\alpha(x-y)P_1(y)\, dy. \end{aligned}$$

Similar considerations for $\partial P_t(y)/\partial t$ lead to

$$B = \int_{|y|>2\sqrt{k}} g_\alpha(x-y)[P_1(y) - P_\varepsilon(y)]\, dy < \int_{|y|>2\sqrt{k}} g_\alpha(x-y)P_1(y)\, dy.$$

Combining the estimates for A and B gives

$$\int_0^1 \left|\frac{\partial g_\alpha(x,t)}{\partial t}\right| dt < \lim_{\varepsilon\to 0} g_\alpha(x,\varepsilon) + g_\alpha(x,1).$$

Again by the Poisson summation formula

$$g_\alpha(x,1) < \left|\sum_m \frac{1}{(1+4\pi^2|m|^2)^{\alpha/2}} e^{-2\pi|m|} e^{im\cdot x}\right| < c(k).$$

Next, let

$$\begin{aligned} f(x, t) &= \sum_m \hat{f}(m) e^{2\pi i m \cdot x - 2\pi |m| t} \\ &= \sum_m \hat{f}_0(m) \frac{1}{(1 + 4\pi^2 |m|^2)^{\alpha/2}} e^{2\pi i m \cdot x - 2\pi |m| t} \\ &= \int_{Q_k} f_0(y) \cdot g_\alpha(x - y, t)\, dy. \end{aligned}$$

It follows that

$$\frac{\partial f}{\partial t}(x, t) = \int_{Q_k} f_0(y) \frac{\partial g_\alpha}{\partial t}(x - y, t)\, dy \tag{2.4}$$

so that

$$\begin{aligned} \int_0^1 \left| \frac{\partial f}{\partial t}(x, t) \right| dt &< \int_{Q_k} |f_0(y)| \int_0^1 \left| \frac{\partial g_\alpha}{\partial t}(x - y, t) \right| dy\, dt \\ &< c(k) \int_{Q_k} |f_0(y)|\, dy + (|f_0| * g_\alpha)(x). \end{aligned} \tag{2.5}$$

We now present the proof of Theorem 1.

Proof of Theorem 1. Let $E_\infty = \{x: \int_0^1 |\partial f(x, t)/\partial t| dt = \infty\}$ and suppose that E_∞ has positive $c_{\alpha,p}$ capacity. Then there exists a measure μ (capacitary distribution) concentrated on $B_\infty = \{x: g_\alpha(x, f_\mu) = 1\} \cap E_\infty$ where $f_\mu(y) = c_{\alpha,p}(E_\infty)[g_\alpha(\mu, y)]^{1/(p-1)}$. We use the notation compatible with §1:

$$\begin{aligned} g_\alpha(\mu, y) &= \int_{Q_k} g_\alpha(x - y)\, d\mu(x), \\ g_\alpha(x, f_\mu) &= \int_{Q_k} g_\alpha(x - y) f_\mu(y)\, dy, \\ c_{\alpha,p}(E_\infty) &= \int f_\mu^p(y)\, dy = c_{\alpha,p}(E_\infty) \int g_\alpha(\mu, y)^{p'}\, dy, \end{aligned}$$

and $\|\mu\| = c_{\alpha,p}(E_\infty) > 0$.

In this case, let $\nu = c_{\alpha,p}^{-1}(E_\infty)\mu$ so that $\|\nu\| = 1$ and $\nu(Q_k - E_\infty) = 0$. Then

$$\begin{aligned} \iint |f_0(y)| g_\alpha(x - y)\, d\nu(x)\, dy &= \int |f_0(y)| \cdot g_\alpha(\nu, y)\, dy \\ &< \|f_0\|_p \cdot \|g_\alpha(\nu, \cdot)\|_{p'} = \|f_0\|_p \cdot c_{\alpha,p}^{-1}(E_\infty). \end{aligned} \tag{2.6}$$

The last inequality follows from the fact that $\|g_\alpha(\mu, \cdot)\|_{p'} < 1$.

We now have that

$$\int_{Q_k} \int_0^1 \left| \frac{\partial f}{\partial t}(x, t) \right| dt\, d\nu(x) = \infty, \tag{2.7}$$

while on the other hand, by (2.5) and (2.6), we have

$$\begin{aligned}\int_{Q_k}\int_0^1\left|\frac{\partial f}{\partial t}(x, t)\right| dt\, d\nu(x) \\ &< c(k)|Q_k|^{1/p'}\|f_0\|_p + \int |f_0|(y) g_\alpha(x-y)\, d\nu(x)\, dy \\ &= \left[c(k)|Q_k|^{1/p'} + c_{\alpha,p}^{-1}(E_\infty)\right]\|f_0\|_p < \infty. \qquad (2.8)\end{aligned}$$

This contradiction completes the proof of Theorem 1.

3. Counterexample. We now give the construction of the counterexample for Theorem 2.

Let Z be a closed set in Q_k such that $c_{\alpha,p}(Z) = 0$. Since $c_{\alpha,p}$ is a capacity, we can approximate $c_{\alpha,p}(Z)$ by $c_{\alpha,p}(K_\rho)$ where $K_\rho = \{x\colon \operatorname{dist}(x, Z) < \rho\}$; that is,

$$c_{\alpha,p}(Z) = \lim_{\rho\to 0} c_{\alpha,p}(K_\rho). \qquad (3.1)$$

For each $\rho > 0$ let μ_ρ and f_ρ be the capacitary distributions for K_ρ with $f_\rho(y)^{p-1} = c_{\alpha,p}(K_\rho)^{p-1} g_\alpha(\mu_\rho, y)$ almost everywhere and $f_\rho * g_\alpha(x) = 1$ almost everywhere with respect to capacity $c_{\alpha,p}$ on K_ρ. Then it follows that

$$\int g_\alpha(x-y)\left\{\int g_\alpha(y-z)\, d\mu_\rho(z)\right\}^{1/(p-1)} dy = c_{\alpha,p}(K_\rho)^{-1} \qquad (3.2)$$

almost everywhere in K_ρ with respect to $c_{\alpha,p}$ capacity. We choose a sequence $\{\mu_{\rho_j}\}$ associated in the above fashion such that $c_{\alpha,p}(K_{\rho_j}) = \|\mu_{\rho_j}\| \to 0$.

Then $\|f_\rho\|_p = c_{\alpha,p}(K_\rho)$. Note that f_ρ is not constant on K_ρ but it has a potential which is equal to one almost everywhere with respect to $c_{\alpha,p}$ capacity. Let

$$f_0^k = \sum_{j=1}^k c_j f_{\rho_j} \quad \text{and} \quad f^k = \sum_{j=1}^k g_\alpha * (c_j f_{\rho_j}). \qquad (3.3)$$

Hence f^k is in $\mathfrak{L}_\alpha^p(Q_k)$. For c_j use $j^{-\delta} c_{\alpha,p}(K_{\rho_j})^{-1}$ with $\delta > 1$ so that we have $\|f_0^k\|_p < \sum_{j=1}^k j^{-\delta} < \infty$ and $\{f_0^k\}$ forms a Cauchy sequence in $L^p(Q_k)$. Let f_0 denote the limit of f_0^k as k tends to infinity. Since f_{ρ_j} is a positive function, f_0 is also the pointwise limit of f_0^k. Furthermore, for almost every x in K_{ρ_k} with respect to $c_{\alpha,p}$ capacity, we have

$$(f_0^k * g_\alpha)(x) > \sum_{j=1}^k j^{-\delta} c_{\alpha,p}(K_{\rho_j})^{-1} (f_{\rho_j} * g_\alpha)(x) = \sum_{j=1}^k j^{-\delta} c_{\alpha,p}(K_{\rho_j})^{-1}. \qquad (3.4)$$

We choose the sequence $\{\rho_j\}$ such that $c_{\alpha,p}(K_{\rho_j}) < j^{-\delta}$. Then in K_{ρ_k}, $(P_t * f_0^k * g_\alpha)$ tends to $\sum_{j=1}^k j^{-\delta} c_{\alpha,p}(K_{\rho_j})^{-1} > k$ as t tends to zero, since $f_0^k * g_\alpha$ is essentially constant in K_{ρ_k}.

Since

$$|f(x, 1) - f(x, t)| < \int_t^1 \left|\frac{\partial f}{\partial s}(x, s)\right| ds \quad \text{and} \quad |f(x, 1)| < c(k)|Q_k|^{1/p'},$$

it will follow that

$$\int_0^1 \left| \frac{\partial f}{\partial t}(x, t) \right| dt = \infty \quad \text{for } x \text{ in } Z \tag{3.5}$$

if we show

$$\lim_{t \to 0} f(x, t) = \infty \quad \text{for } x \text{ in } Z. \tag{3.6}$$

But for x in Z we have

$$\lim_{t \to \infty} \inf f(x, t) > \lim_{t \to \infty} \inf\left(P_t * f_0^k * g_\alpha\right) > k$$

for all k. This concludes the construction of Theorem 2.

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