PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 2, November 1979

COREFLECTORS NOT PRESERVING THE INTERVAL AND BAIRE PARTITIONS OF UNIFORM SPACES

G. TASHJIAN[1] AND J. VILÍMOVSKÝ

ABSTRACT. Values of the compact interval and other spaces under coreflectors in the category of uniform spaces are studied. It is shown that any coreflector which changes the usual uniformity of the interval produces a new uniformity which contains all finite Baire partitions of the interval.

Introduction. Coreflectors in the category of uniform spaces which do not preserve the unit interval are studied. The main result is that there exists a largest coreflective subcategory of uniform spaces not containing the closed unit interval, and this is the category Ba-fine$|_I$ of all uniform spaces in which each finite Baire partition is a uniform cover. (This category has been studied by others, as in [2] and [3].) The result is obtained by analyzing the action of coreflectors on the convergent sequence. Some new characterizations of the class Ba-fine$|_I$ are also given.

The main result was motivated by a problem about cartesian-closed subcategories of uniform spaces, and it is needed in [6].

Preliminaries. Throughout the paper it is assumed that all spaces belong to the category Unif of Hausdorff uniform spaces. All subcategories of Unif are assumed to be full, isomorphism-closed, and to contain a nonempty space.

For two spaces X and Y in Unif having the same underlying set, the notation $Y < X$ is used if the identity map i: $Y \to X$ is uniformly continuous, that is, if Y is finer than X. If $Y < X$, denote by $X - Y$ the class of all uniform spaces Z for which each uniformly continuous map $Z \to X$ remains uniformly continuous into Y. It is easy to verify that the class $X - Y$ is closed under the formation of sums and quotients and hence it forms a coreflective subcategory of Unif. Furthermore, if X is an injective uniform space, then $X - Y$ is hereditary (closed under subspace formation).

Let I be the closed unit interval [0, 1] with the usual metric uniformity, and let S be the convergent sequence $\{1/n\colon n < \omega\} \cup \{0\}$ with uniformity inherited from I. Let I' be the space on I whose uniformity is generated by the cover $\{\{0\}, I - \{0\}\}$ in addition to the usual open covers of I. Let S' be the space on S with uniformity inherited from I'. Finally, let S'' be the space

Received by the editors May 2, 1978 and, in revised form, June 30, 1978.
AMS (MOS) subject classifications (1970). Primary 54E15, 54B25.
Key words and phrases. Uniform spaces, coreflection, compact interval, Baire set.

[1]Written while the first author was a National Academy of Sciences Exchange Scientist in Czechoslovakia, 1977–78.

0002-9939/79/0000-0517/$02.75

on S whose uniformity consists of all finite partitions of S. Then $I' < I$ and $S'' < S' < S$. Also, S' is uniformly isomorphic to any Cauchy sequence.

Coreflections not preserving S or I. We begin by considering the values of the spaces defined above under coreflectors in Unif. In general, the action of any such functor on a space preserves the underlying set and refines the original uniformity, by a result in [5].

First, we show that there exists a largest coreflective subcategory of Unif not containing S.

LEMMA 1. *Let F be a coreflector in* Unif. *If $FS \neq S$, then $FS < S''$.*

PROOF. Since S is compact, the topology and proximity of FS are finer than those of S. Hence, there exist subsets A and B of S which are near in S and far in FS. These sets are infinite, since they are near in S, so each set is near $\{0\}$ in S. However, since A and B are far in FS, one of them, say A, is far from $\{0\}$ in FS.

Now let $\{C, D\}$ be any partition of S. Define a function f: $S \to S$ such that $f(x) = 0$ for each $x \in C$, $f(x) \in A$ for each $x \in D$, and f is one-to-one on D. Such a map exists since A is infinite. Furthermore, it is easy to check that f is uniformly continuous. So, f: $FS \to FS$ is also uniformly continuous, hence proximally continuous. Since A and $\{0\}$ are far in the range FS, the sets $D = f^{-1}(A)$ and $C = f^{-1}(\{0\})$ are far in the domain FS. Hence, the partition $\{C, D\}$ must be a uniform cover of FS.

Therefore, all two-element partitions of FS are uniform covers, so all finite partititons of FS are uniform covers and $FS < S''$.

COROLLARY 1. *Let $\mathcal{C}$ be a coreflective subcategory of* Unif. *Then either $S \in \mathcal{C}$ or $\mathcal{C} \subseteq S - S''$, so $S - S''$ is the largest coreflective subcategory of* Unif *not containing S.*

COROLLARY 2. *$S - S' = S - S''$. If F is the coreflector associated with this subcategory, then $FS = S''$.*

PROOF. The first statement follows immediately from Corollary 1. For the second, we have $FS < S''$ by Lemma 1. Also, $S'' < FS$ since S'' is the value of S under the proximally discrete coreflector. Hence, $FS = S''$.

If $\mathcal{C}$ is a class of spaces, let her($\mathcal{C}$) denote the class of all spaces which are hereditarily in $\mathcal{C}$. Let sub($\mathcal{C}$) be the collection of all subspaces of members of $\mathcal{C}$. Then her($\mathcal{C}$) $\subseteq \mathcal{C} \subseteq$ sub($\mathcal{C}$). Furthermore, if $\mathcal{C}$ is coreflective, then sub($\mathcal{C}$) is also coreflective ([7]), but her($\mathcal{C}$) need not be coreflective.

THEOREM 1. *Let $X \in$* Unif. *The following are equivalent*:

(a) $X \in$ her$(S - S')$.

(b) *If $\{A_n: 0 \leqslant n < \omega\}$ is a countable family of subsets of X such that for every $m \geqslant 1$, A_m is far from $\bigcup_{n \neq m} A_n$, then A_0 is far from $\bigcup_{n>0} A_n$.*

(c) $X \in I - I'$.

PROOF. (a) $\Rightarrow$ (b). Let $X \in \text{her}(S - S')$ and let $\{A_n\}$ be a family of subsets of X satisfying the hypothesis of (b). Let $A = \bigcup_{n<\omega} A_n$, with uniformity inherited from X. Then $A \in S - S'$.

Now define a function $f\colon A \to S$ by $f(x) = 1/n$ if $x \in A_n$ and $n \neq 0$, and $f(x) = 0$ if $x \in A_0$. Then f is uniformly continuous since A_m is far from $\bigcup_{n\neq m} A_n$, for $m \geqslant 1$.

Therefore, $f\colon A \to S'$ is uniformly continuous. Since $\{0\}$ is far from $\{1/n\colon n < \omega\}$ in S', A_0 is far from $\bigcup_{n>0} A_n$ in A.

(b) $\Rightarrow$ (c). Let X satisfy (b), and let $f\colon X \to I$ be uniformly continuous. Let $A_0 = \{0\}$ and $A_n = [1/2n, 1/(2n-1)]$, for $n \geqslant 1$, in I. Then $\{A_n\colon 0 \leqslant n < \omega\}$ is a family of subsets of I satisfying the hypothesis of (b). Hence, the same is true in X for the family $\{f^{-1}(A_n)\colon 0 \leqslant n < \omega\}$. Therefore, $f^{-1}(\{0\})$ is far from $\bigcup_{n>0} f^{-1}(A_n)$, by (b).

Similarly, we may define $B_0 = \{0\}$, $B_n = [1/(2n+1), 1/2n]$ for $n \geqslant 1$ and repeat the above argument for this family to conclude that $f^{-1}(\{0\})$ is far from $\bigcup_{n>0} f^{-1}(B_n)$.

Therefore, $f^{-1}(\{0\})$ is far from the union of $\bigcup_{n>0} f^{-1}(A_n)$ and $\bigcup_{n>0} f^{-1}(B_n)$, and this union is $f^{-1}(I - \{0\})$. It follows that $f\colon I \to I'$ is uniformly continuous, so $X \in I - I'$.

(c) $\Rightarrow$ (a). Let $X \in I - I'$. Using the natural embeddings of S and S' into I and I', respectively, one may easily check that if $f\colon X \to S$ is uniformly continuous, then so is $f\colon X \to S'$. Hence, $X \in S - S'$. Now I is an injective space, so $I - I'$ is hereditary. Therefore $X \in \text{her}(S - S')$.

REMARKS. (1) Since $I - I'$ is coreflective, it follows from the theorem that $\text{her}(S - S')$ is coreflective.

(2) Condition (b) shows that the properties given in (a) and (c) depend only on proximity structure.

If $X \in \text{Unif}$, let $\text{Coz}(X)$ be the collection of all cozero sets of uniformly continuous real-valued functions on X, let $Z(X)$ be the collection of all zero sets in X, and let $\text{Ba}(X)$ be the Baire σ-algebra on X generated by $\text{Coz}(X)$. A function $f\colon X \to Y$ is Baire-measurable, (cozero-measurable) if $f^{-1}(A)$ is a Baire (cozero) set in X whenever A is a Baire (cozero) set in Y.

Define the class $\text{Ba-fine}|_I$ ($\text{Coz-fine}|_I$) to consist of all uniform spaces X such that any Baire (cozero)-measurable map from X into I is uniformly continuous. The members of $\text{Coz-fine}|_I$ are sometimes called Alexandroff spaces. These classes are mentioned in [2] and [3], and it is known that they are coreflective subcategories of Unif.

THEOREM 2. *Let* $X \in \text{Unif}$. *The following are equivalent to conditions* (a)–(c) *in Theorem* 1:

(d) $X \in \textit{Ba-fine}|_I$.

(e) *Any finite Baire partition of* X *is a uniform cover.*

(f) $X \in \textit{Coz-fine}|_I$ *and* $\text{Coz}(X) = Z(X) = \text{Ba}(X)$.

(g) *If f: $X \to I$ is the pointwise limit of a sequence of uniformly continuous maps, then f is uniformly continuous.*

PROOF. The equivalence of conditions (d)–(g) is known from [2] and [3].

We will show that (e) is equivalent to condition (c) of Theorem 1.

Suppose X satisfies (e). If f: $X \to I$ is uniformly continuous, then f: $X \to I'$ is uniformly continuous since $f^{-1}(\{\{0\}, I - \{0\}\})$ is a Baire partition of X. Hence $X \in I - I'$.

Now suppose $X \in I - I'$. If f: $X \to I$ is uniformly continuous, then f: $X \to I'$ is also, so $\{f^{-1}(\{0\}), f^{-1}(I - \{0\})\}$ is a uniform cover of X. Hence, the zero set $f^{-1}(\{0\})$ is also a cozero set in X, so

$$Z(X) = \mathrm{Coz}(X) = \mathrm{Ba}(X).$$

Since any two-element Baire partition of X has the above form, it must be a uniform cover. It follows that any finite Baire partition of X is a uniform cover, so X satisfies (e).

THEOREM 3. *Ba-fine$|_I$ is the largest coreflective subcategory of* Unif *not containing I.*

PROOF. Let $\mathcal{C}$ be a coreflective subcategory such that $I \notin \mathcal{C}$. Then $I \notin \mathrm{sub}(\mathcal{C})$. Otherwise, there exists $X \in \mathcal{C}$ such that $I \subseteq X$. Now I is injective, so it is a retract and hence a quotient, of any space which contains it. Then I would belong to $\mathcal{C}$.

Now $\mathcal{C} \subseteq \mathrm{sub}(\mathcal{C})$, $\mathrm{sub}(\mathcal{C})$ is coreflective, and $I \notin \mathrm{sub}(\mathcal{C})$. Let F be the coreflector onto $\mathrm{sub}(\mathcal{C})$. Then $FI \neq I$ and F preserves subspaces, since $\mathrm{sub}(\mathcal{C})$ is hereditary. Since $FI < I$ and I is compact, the topology of FI is finer and that of I, so there exists a convergent sequence S in I which does not converge in FI. Hence $FS \neq S$, so by Corollary 1, $\mathrm{sub}(\mathcal{C}) \subseteq S - S'$. Hence, $\mathcal{C} \subseteq \mathrm{her}(S - S')$, so by Theorem 2, $\mathcal{C} \subseteq \text{Baire-fine}|_I$.

REMARKS. (1) As we mentioned earlier, if $\mathcal{C}$ is coreflective, $\mathrm{her}(\mathcal{C})$ need not be. For example, in [1] it is shown that $\mathrm{her}(\text{Coz-fine}|_I)$ is not coreflective. Also, $I \notin \mathrm{her}(\text{Coz-fine}|_I)$ since its subspace $\{1/n: n < \omega\}$ does not belong to $\text{Coz-fine}|_I$. Therefore, from the results above, $\text{Ba-fine}|_I$ is the largest coreflective subcategory in $\mathrm{her}(\text{Coz-fine}|_I)$.

(2) We have seen that, for some spaces X, there exists a largest coreflective subcategory not containing X. This is true for exactly those spaces X satisfying the condition: whenever X is inductively generated by a family $\mathcal{S}$, then there exists a space $Y \in \mathcal{S}$ which inductively generates X. This fails for many spaces.

An immediate consequence of Theorem 3 is the following:

COROLLARY 3. *$X \notin$ Ba-fine$|_I$ if and only if I is a quotient of a sum of copies of X.*

Let $\mathfrak{M}$ be the collection of all metrizable uniform spaces, and let $\mathrm{co}(\mathfrak{M})$ be its coreflective hull in Unif. We will show that any nonuniformly discrete

space in co($\mathfrak{M}$) does not belong to Baire-fine$|_I$; hence it inductively generates I.

Let D_2 be the countable metrizable space defined as follows: $D_2 = \{x_n\} \cup \{y_n\}$, $n \in N$ with metric d satisfying

$$d(x_n, x_m) = d(x_n, y_m) = d(y_n, y_m) = 1$$

if $n \neq m$, and $d(x_n, y_n) = 1/n$ for $n \in N$.

It is known (cf. e.g. [4]) that any metrizable uniform space is a uniform quotient of a sum of copies of D_2, hence co($\mathfrak{M}$) = co($\{D_2\}$).

Also, any precompact metrizable space X is a quotient of the sum of its Cauchy sequences, for if any function on X fails to be uniformly continuous, there is a Cauchy sequence in X on which that function is not uniformly continuous.

THEOREM 4. *The unit interval may be inductively generated by any nondiscrete member of* co($\mathfrak{M}$).

PROOF. Let X be a nondiscrete space in co($\mathfrak{M}$). Then X is inductively generated by D_2.

First, there exists a uniformly continuous map f: $D_2 \to X$ such that $f(x_n) \neq f(y_n)$ for infinitely many n. Otherwise, $f(x_n) = f(y_n)$ for all but finitely many n, so any function g on X would have uniformly continuous composition with f. If this were true for all f: $D_2 \to X$, then X would be uniformly discrete.

Let S be an infinite subset of N such that $f(x_n) \neq f(y_n)$ for all $n \in S$. Now $\{x_n: n \in S\} \cup \{y_n: n \in S\}$ is a subspace of D_2 isomorphic to D_2, so we may assume that $f(x_n) \neq f(y_n)$ for all $n \in N$.

The range of this function f must be infinite. If not, there exist an infinite subset T of N and points p, q in X such that $f(x_n) = p$ and $f(y_n) = q$ for all $n \in T$. Now the infinite sets $\{x_n: n \in T\}$ and $\{y_n: n \in T\}$ are near in the proximity of D_2, so their images under f must be near in X. Hence, $p = q$, contradicting $f(x_n) \neq f(y_n)$.

This shows that if $X \in$ co($\mathfrak{M}$), then there exists a uniformly continuous map f: $D_2 \to X$ with infinite range such that $f(x_n) \neq f(y_n)$ for all $n \in N$, provided X is nondiscrete. Let $D_2 = A \cup B$, where $A = \{x_n: n \in N\}$ and $B = \{y_n: n \in N\}$.

Case 1. Either $f(A)$ or $f(B)$ is finite. Then the infinite set would contain a Cauchy sequence C converging to a point in the finite set, by uniform continuity of f. The interval I is inductively generated by C since I is compact and metrizable.

Furthermore, any map from C into I extends uniformly to X, since I is injective in Unif. Hence, I is inductively generated by X.

Case 2. Both $f(A)$ and $f(B)$ are infinite. We may assume without loss of generality that the restrictions $f|_A$ and $f|_B$ are one-to-one. (Otherwise, we may pass to appropriate subsets of A and B and apply Case 1.) Then it is possible to define by induction a subsequence $S = \{n_k: k \in N\}$ of N such that

$$\{f(x_m): m \in S\} \cap \{f(y_m): m \in S\} = \varnothing.$$

Identifying $\{x_m: m \in S\} \cup \{y_m: m \in S\}$ with D_2, we now have a uniformly continuous one-to-one map $f: D_2 \to X$.

We will show that $X \notin \text{Ba-fine}|_I$. Since D_2 is countable, there is a uniform pseudometric r on X such that $r(f(a), f(b)) > 0$ for all $a \neq b$ in D_2. Let P_n and Q_n be the r-spheres of radius 0 about $f(x_n)$ and $f(y_n)$, respectively, for $n \in N$. Then the sets $P = \bigcup \{P_n: n \in N\}$ and $Q = \bigcup \{Q_n: n \in N\}$ are disjoint Baire sets in X, so $\{P, Q, X - (P \cup Q)\}$ is a finite Baire partition of X. However, the inverse image of this partition under f is not a uniform cover of D_2, so the partition cannot be a uniform cover of X. Hence $X \notin \text{Ba-fine}|_I$, so by Corollary 3, I is inductively generated by X.

REMARK. If M is a nondiscrete metrizable space, then M contains either a Cauchy sequence or a copy of D_2 as a uniform subspace, by the considerations in the preceding proof.

COROLLARY 4. *Let $\mathcal{C}$ be a coreflective subcategory of* Unif. *The following are equivalent*:

(a) $I \notin \mathcal{C}$.

(b) $\mathcal{C} \subseteq$ *Ba-fine*$|_I$.

(c) $\mathcal{C}$ *contains no nondiscrete metrizable spaces.*

(d) $\mathcal{C}$ *contains no nondiscrete member of* $\text{co}(\mathfrak{M})$.

We conclude with a special case. Let R be the real line with the usual metric uniformity.

DEFINITION. The class $\text{Ba-fine}|_R$ is the collection of all spaces X for which each Baire-measurable map into R is uniformly continuous.

It is known from [3] that $\text{Ba-fine}|_R$ is a coreflective subcategory of Unif and that $X \in \text{Ba-fine}|_R$ if and only if one of the following holds:

(1) Each countable Baire partition of X is a uniform cover.

(2) If M is a separable metrizable space and $f: X \to M$ is Baire-measurable, then f is uniformly continuous.

(3) If M is a separable metrizable space, then any pointwise limit of asequence of uniformly continuous maps from X into M is uniformly continuous.

COROLLARY 5. *Let F*: Unif $\to \mathcal{C}$ *be a coreflection. The following are equivalent*:

(1) *FI is not precompact.*

(2) $\mathcal{C} \subseteq$ *Ba-fine*$|_R$.

(3) $\mathcal{C}$ *contains no infinite precompact space.*

PROOF. (3) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2). Since FI is not precompact, it contains an infinite uniformly discrete subspace $B = \{x_n: n < \omega\}$, which we may assume is Cauchy in I.

Now let $X \in \mathcal{C}$ and let $\{Q_n: n < \omega\}$ be a countable Baire partition of X.

Define a function $f\colon X \to I$ by $f(x) = x_n$ for $x \in Q_n$. Then f is uniformly continuous because the finite Baire partitions of X are uniform, by Theorem 3, and $\{x_n\}$ is Cauchy in I. Therefore, $f\colon X \to FI$ is uniformly continuous, so $\{Q_n\}$ is a discrete family in X. Hence, $\{Q_n\}$ is a uniform cover of X, so $X \in \text{Ba-fine}|_R$.

(2) $\Rightarrow$ (3). This follows from the fact that any infinite space has an infinite Baire partition; and such a partition cannot be a uniform cover if the space is precompact.

References

1. Z. Frolík, *Four functors into paved spaces*, Seminar Uniform Spaces 1973-74, MŮ ČSAV Prague 1975, pp. 27–72.

2. _____, *Measurable uniform spaces*, Pacific J. Math. **55** (1974), 93–105.

3. A. W. Hager, *Three classes of uniform spaces*, (Proc. Third Prague Topological Symposium, 1971), Academia, Prague, 1972, pp. 159–164.

4. J. Isbell, *Uniforms paces*, Math. Surveys, no. 12, Amer. Math. Soc., Providence, R. I., 1964.

5. J. F. Kennison, *Reflective functors in general topology and elsewhere*, Trans. Amer. Math. Soc. **118** (1965), 303–315.

6. M. D. Rice and G. Tashjian, *Cartesian-closed coreflective subcategories of uniform spaces*, Trans. Amer. Math. Soc. (to appear).

7. J. Vilímovský, *Generation of coreflections in categories*, Comment. Math. Univ. Carolinae **14** (1973), 305–323.

Mathematics Institute, Czechoslovak Academy of Sciences, Žitná 25, 115 67 Prague 1, Czechoslovakia (Current address of J. Vilímovský)

Current address (G. Tashjian): Department of Mathematics, Clark University, Worcester, Massachusetts 01610