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ON REMOTE POINTS IN $\upsilon X - X$

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ABSTRACT. Under a certain set-theoretic assumption, a question of E. K. van Douwen is solved. More precisely, if the cellularity $c(X)$ of a space X is nonmeasurable, then $\upsilon X - X$ contains no remote point of X.

All spaces considered here are Tychonoff. For a space X, βX is the Stone-Čech compactification and υX is the Hewitt realcompactification of X. A point $p \in \beta X - X$ is said to be remote if $p \notin \mathrm{Cl}_{\beta X} D$ for every nowhere dense subset D of X. In [1], Eric K. van Douwen discussed fully the theory of remote points. He raised the following question: Can $\upsilon X - X$ contain a remote point of X?

In this note we shall show that under a certain set-theoretic assumption $\upsilon X - X$ cannot contain a remote point of X.

A cardinal $\mathfrak{m}$ is called measurable if a set X of cardinality $\mathfrak{m}$ admits a $\{0, 1\}$-valued measure μ such that $\nu(X) = 1$, and $\mu(\{x\}) = 0$ for every $x \in X$. The proposition that no measurable cardinal exists is known to be consistent with ZFC (see [2]). Let us recall a cardinal function given in [3]. The cellularity of a space X is $c(X) = \sup\{|\mathfrak{U}|: \mathfrak{U}$ is a family of pairwise disjoint nonempty open subsets of $X\}$.

THEOREM. *For a space X, if $c(X)$ is nonmeasurable, then $\upsilon X - X$ contains no remote point of X.*

PROOF. Let p be a point of $\upsilon X - X$. Let $\mathfrak{U}$ be a maximal collection of pairwise disjoint nonempty open subsets of X such that $p \notin \mathrm{Cl}_{\beta X} U$ for each $U \in \mathfrak{U}$. Then $|\mathfrak{U}|$ is nonmeasurable since $|\mathfrak{U}| \leqslant c(X)$. Let $D = X - \bigcup \mathfrak{U}$, where $\bigcup \mathfrak{U} = \bigcup \{U: U \in \mathfrak{U}\}$. Then D is a nowhere dense subset of X. Hence to see that p is not a remote point of X it suffices to show that $p \in \mathrm{Cl}_{\beta X} D$. Assume that $p \notin \mathrm{Cl}_{\beta X} D$. Let

$$\mathfrak{F} = \{\mathfrak{V}: \mathfrak{V} \subset \mathfrak{U}, p \in \mathrm{Ex}_X(\bigcup \mathfrak{V})\},$$

where $\mathrm{Ex}_X U = \beta X - \mathrm{Cl}_{\beta X}(X - U)$ for every open subset U of X. We shall show that $\mathfrak{F}$ is an ultrafilter on $\mathfrak{U}$ with the countable intersection property. Since

$$\mathrm{Ex}_X(\bigcap \{U_i: i = 0, 1, \ldots, n\}) = \bigcap \{\mathrm{Ex}_X U_i: i = 0, 1, \ldots, n\}$$

for each finite collection $\{U_i: i = 0, 1, \ldots, n\}$ of open subsets of X (see [1]),

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$\mathfrak{F}$ has the finite intersection property. Let $\mathcal{V}$ be a subcollection of $\mathcal{U}$. Since

$$\begin{aligned}\beta X &= \mathrm{Cl}_{\beta X}(\bigcup \mathcal{V}) \cup \mathrm{Cl}_{\beta X}(\bigcup(\mathcal{U} - \mathcal{V}))\\ &= (\mathrm{Bd}_{\beta X}(\mathrm{Ex}_X(\bigcup \mathcal{V})) \cup \mathrm{Ex}_X(\bigcup \mathcal{V}))\\ &\quad \cup (\mathrm{Bd}_{\beta X}(\mathrm{Ex}_X(\bigcup(\mathcal{U} - \mathcal{V}))) \cup \mathrm{Ex}_X(\bigcup(\mathcal{U} - \mathcal{V})))\\ &= (\mathrm{Cl}_{\beta X}(\mathrm{Bd}_X(\bigcup \mathcal{V})) \cup \mathrm{Ex}_X(\bigcup \mathcal{V}))\\ &\quad \cup (\mathrm{Cl}_{\beta X}(\mathrm{Bd}_X(\bigcup(\mathcal{U} - \mathcal{V}))) \cup \mathrm{Ex}_X(\bigcup(\mathcal{U} - \mathcal{V})))\\ &= \mathrm{Cl}_{\beta X} D \cup \mathrm{Ex}_X(\bigcup \mathcal{V}) \cup \mathrm{Ex}_X(\bigcup(\mathcal{U} - \mathcal{V}))\end{aligned}$$

and $p \notin \mathrm{Cl}_{\beta X} D$, it is obvious that $p \in \mathrm{Ex}_X(\bigcup \mathcal{V})$ or $p \in \mathrm{Ex}_X(\bigcup(\mathcal{U} - \mathcal{V}))$. This implies that $\mathfrak{F}$ is an ultrafilter on $\mathcal{U}$. Let us show that $\mathfrak{F}$ has the countable intersection property. Assume that there is a sequence $\{\mathcal{V}_i : i < \omega\}$ of elements of $\mathfrak{F}$ such that

$$\mathcal{V} = \bigcap \{\mathcal{V}_i : i < \omega\} \notin \mathfrak{F}.$$

Then $p \notin \mathrm{Cl}_{\beta X}(\bigcup \mathcal{V})$ since

$$\mathrm{Cl}_{\beta X}(\bigcup \mathcal{V}) = \mathrm{Ex}_X(\bigcup \mathcal{V}) \cup \mathrm{Bd}_{\beta X}(\mathrm{Ex}_X(\bigcup \mathcal{V})),$$

$p \notin \mathrm{Ex}_X(\bigcup \mathcal{V})$ and $p \notin \mathrm{Bd}_{\beta X}(\mathrm{Ex}_X(\bigcup \mathcal{V})) = \mathrm{Cl}_{\beta X}(\mathrm{Bd}_X(\bigcup \mathcal{V})) \subset \mathrm{Cl}_{\beta X} D$. Hence there is a zero-set Z_ω of βX such that $p \in Z_\omega$ and $Z_\omega \cap \mathrm{Cl}_{\beta X}(\bigcup \mathcal{V}) = \varnothing$. On the other hand, for each $i < \omega$ there is a zero-set Z_i of βX such that $p \in Z_i$ and $Z_i \subset \mathrm{Ex}_X(\bigcup \mathcal{V}_i)$. Now, let $Z = \bigcap \{Z_i : i \leqslant \omega\}$. Then Z is a zero-set of βX which contains p. Since

$$\bigcup \mathcal{V} = \bigcup(\bigcap \{\mathcal{V}_i : i < \omega\}) = \bigcap \{\bigcup \mathcal{V}_i : i < \omega\},$$

$$Z \cap X \subset (\bigcap \{\bigcup \mathcal{V}_i : i < \omega\}) \cap (X - \mathrm{Cl}_X(\bigcup \mathcal{V})) = \varnothing.$$

But this is a contradiction since $p \in \upsilon X$. Hence $\mathfrak{F}$ has the countable intersection property. But, since $\{U\} \notin \mathfrak{F}$ for each $U \in \mathcal{U}$, this contradicts the fact that $|\mathcal{U}|$ is nonmeasurable (see [2]).

COROLLARY 1. *Assume that every cardinal is nonmeasurable. Then $\upsilon X - X$ contains no remote point of X for any space X.*

COROLLARY 2. *Assume that every cardinal is nonmeasurable. If X has a remote point, then X is not pseudocompact.*

Corollary 2 shows that nonpseudocompactness is essential to have a remote point.

REMARK. The converse of the above Theorem is not true (i.e. the nonmeasurability of $c(X)$ need not be implied by the fact that $\upsilon X - X$ contains no remote point of X). However the nonmeasurability of $c(X)$ cannot be dropped in the Theorem. In fact, let M be a discrete space of measurable cardinality. Then $\upsilon M - M$ is nonempty, and every point of $\upsilon M - M$ is a remote point of M.

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