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PSEUDO LATTICE PROPERTIES OF THE STAR-ORTHOGONAL PARTIAL ORDERING FOR STAR-REGULAR RINGS

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ABSTRACT. It is shown that a star-regular ring R forms a pseudo upper semilattice under the star-orthogonal partial ordering. That is, for every a, b in R, the set $\{c | c > a, c > b\}$ is nonempty if and only if $a \vee b$ exists in R, in which case

$$a \vee b = a + (1 - aa^{\dagger})bb^{*}[(1 - a^{\dagger}a)b^{*}]^{\dagger}.$$

1. Introduction. In two recent papers ([2], [3]), Drazin introduced the star-orthogonal partial ordering

$$a < b \Leftrightarrow a^{*}a = a^{*}b \quad \text{and} \quad aa^{*} = ba^{*} \tag{1}$$

for *proper*-star-semigroups $(S, {}^{*})$, for which the involution $(\cdot)^{*}: S \to S$ satisfies, in addition to the usual two conditions (i) $(a^{*})^{*} = a$, (ii) $(ab)^{*} = b^{*}a^{*}$, the "proper" condition (iii) $a^{*}a = a^{*}b = b^{*}a = b^{*}b \Rightarrow a = b$. For a ring R, the condition (iv) $(a + b)^{*} = a^{*} + b^{*}$ is added, and (iii) is easily seen to be equivalent to the traditional star cancellation law

$$a^{*}a = 0 \Rightarrow a = 0. \tag{2}$$

It was subsequently shown by Hartwig and Drazin [6] that the algebra $\mathbf{C}_{n\times n}$ of $n \times n$ complex matrices forms a *lower* semilattice under the partial ordering (1), which means that $a \wedge b = \sup\{c | c < a, c < b\}$ exists in $\mathbf{C}_{n\times n}$ for all a, b in $\mathbf{C}_{n\times n}$. Because invertible elements are obviously *maximal* elements under $<$, the join $a \vee b = \inf\{c | c > a, c > b\}$ will in general not exist, because the set $\{c | c > a, c > b\}$ may be empty.

The purpose of this note is to prove that if R is a star-regular ring, then R forms a pseudo upper semilattice, that is $a \vee b$ will exist precisely when $\{c | c > a, c > b\}$ is nonempty. An element $a \in S$ is called *regular* if $a \in aSa$, and $*$-regular if aa^{*} and $a^{*}a$ are both regular. It is well known, from [8], that $a \in S$ is star-regular exactly when there is a, necessarily unique, solution to the equations:

$$axa = a, \qquad xax = x, \qquad (ax)^{*} = ax, \qquad (xa)^{*} = xa.$$

This solution $a^{\dagger}$ is known as the Moore-Penrose inverse of a. A ring is called (star) regular when every element $a \in R$ is (star) regular. It should be noted that R is $*$-regular precisely when R is regular and the involution is proper.

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2. Main results. We shall now prove our main local result, from which the global result for star-regular rings obviously follows.

THEOREM 1. *Let R be a ring with involution $(\cdot)^*$, and let a, b be elements of R such that a, b, $(1 - aa^\dagger)b$, and $b(1 - a^\dagger a)$ are star-regular. Then $\{c | c \geq a, c \geq b\}$ is nonempty if and only if*

$$\text{(i)}\quad b(b^* - a^*)a = 0 = a(b^* - a^*)b,$$

$$\text{(ii)}\quad b(b^* - a^*) \in b(1 - a^\dagger a)R,$$

$$\text{(iii)}\quad (b^* - a^*)b \in R(1 - aa^\dagger)b. \tag{3}$$

In which case $a \vee b$ exists and is given by

$$a \vee b = a + (1 - aa^\dagger)bb^*\left[(1 - a^\dagger a)b^*\right]^\dagger. \tag{4}$$

PROOF. Suppose that $c \geq a$, $c \geq b$ for some $c \in R$, that is

$$a^*(a - c) = (a - c)a^* = b^*(b - c) = (b - c)b^* = 0. \tag{5a}$$

Since a and b are $*$-regular, $a^\dagger$ and $b^\dagger$ exist and hence (5a) may be rewritten as in [3]:

$$aa^\dagger c = a = ca^\dagger a, \qquad bb^\dagger c = b = cc^\dagger b. \tag{5b}$$

Thus

$$a^\dagger b = a^\dagger cb^\dagger b = a^\dagger(aa^\dagger cb^\dagger b) = a^\dagger ab^\dagger b, \qquad b^\dagger a = b^\dagger ba^\dagger a. \tag{6}$$

Symmetry now yields two more such results. From (6), $a^*b = a^*ab^\dagger b$, which shows that $b^\dagger ba^*a = b^*a$ and hence that $ba^*a = bb^*a$. By symmetry $aa^*b = ab^*b$, so that (3i) follows. Next, let $u = (1 - aa^\dagger)b$ and $v = b(1 - a^\dagger a)$, and consider $b^*c = b^*b$. Post multiplication by $(1 - a^\dagger a)$ yields

$$b^*b(1 - a^\dagger a) = b^*c(1 - a^\dagger a) = b^*(1 - aa^\dagger)c,$$

that is

$$u^*c = b^*v. \tag{7}$$

Similarly $(1 - aa^\dagger)cb^* = c(1 - a^\dagger a)b^* = (1 - aa^\dagger)bb^*$ yields

$$cv^* = ub^*. \tag{8}$$

The assumed consistency of (7) and (8) ensures that $u^*u^{*\dagger}b^*v = b^*v$ and $ub^*v^{*\dagger}v^* = ub^*$, while the elimination of c gives

$$u^*ub = u^*cv^* = b^*vv^*. \tag{9}$$

Now

$$u^\dagger ub^*v = b^*v \Leftrightarrow v^*b = v^*bu^\dagger u \Leftrightarrow v^*b \in Ru \Leftrightarrow v^\dagger b = v^\dagger bu^\dagger u, \tag{10a}$$

where

$$v^*b = (1 - a^\dagger a)b^*b = b^*b - a^\dagger ab^*b = b^*b - a^*bb^\dagger b = (b^* - a^*)b,$$

from which (3iii) follows.

Similarly

$$ub^*vv^\dagger = ub^* \Leftrightarrow vv^\dagger bu^* = bu^* \Leftrightarrow bu^* \in vR \Leftrightarrow vv^\dagger bu^\dagger = bu^\dagger, \tag{10b}$$

where $bu^* = bb^*(1 - aa^\dagger) = bb^* - ba^*$. This completes the proof of the necessity of (3).

Suppose now that (3i), (3ii) and (3iii) hold. We shall first demonstrate that $\{r|r \geqslant a, r \geqslant b\}$ is nonempty.

First observe that a particular solution to the equations (7) and (8) alone is given by $u^{*\dagger}b^*v$. To obtain a solution to $r \geqslant a, r \geqslant b$, all we have to do is add element a to $w = u^{*\dagger}b^*v$.

Indeed, since $a^\dagger u = 0 = u^\dagger a = va^\dagger = av^\dagger$, we have $a^\dagger w = wa^\dagger = 0$, and

$$aa^\dagger(a + w) = a = (a + w)a^\dagger a \quad \text{or} \quad a \leqslant a + w.$$

Next, consider $bb^\dagger(a + w) = ba^\dagger a + bb^\dagger u^{*\dagger}b^*v$, where we used (3i), and recall that *always*:

$$\begin{aligned} &u^*u = u^*b = b^*u, && vv^* = vb^* = bv^*, && ub^\dagger b = u, \\ &u^\dagger u = u^\dagger b, && vv^\dagger = bv^\dagger, && bb^\dagger v = v, \\ &u = uu^\dagger b, && v = bv^\dagger v. && \end{aligned} \tag{11}$$

Hence, $bb^\dagger u^{*\dagger}b^*v = (u^\dagger bb^\dagger)^*b^*v$ which by (11) becomes $(u^\dagger ub^\dagger)^*b^*v = b^{*\dagger}(u^\dagger ub^*v)$. Using (10a) this reduces to $b^{*\dagger}b^*v = bb^\dagger v$, and hence by (11) equals $v = b - ba^\dagger a$. Substituting this in the above we see that $bb^\dagger(a + w) = b$. Similarly, with aid of (10b), (3i) and (11), $(a + w)b^\dagger b = aa^\dagger b + u^{*\dagger}b^*vb^\dagger b = aa^\dagger b + u = b$, and thus $a \leqslant a + w$, $b \leqslant a + w$, as desired. In conclusion let us prove that $a + w$ is in fact equal to $a \vee b$. In order to do this, let us first verify that $w^\dagger$ exists and that

$$(a + w)^\dagger = a^\dagger + w^\dagger. \tag{12}$$

The details are essential since we shall also need the expressions for $(a + w)(a + w)^\dagger$ and $(a + w)^\dagger(a + w)$. Again, since $a^*w = 0 = wa^*$, it follows by a result of Hestenes [7] that (12) holds and that in addition:

$$(a + w)(a + w)^\dagger = aa^\dagger + ww^\dagger, \quad (a + w)^\dagger(a + w) = a^\dagger a + w^\dagger w, \tag{13}$$

provided $w^\dagger$ exists. Let us now verify that $x = v^\dagger b^{*\dagger}u^* = w^\dagger$.

Indeed, $wx = u^{*\dagger}b^*vv^\dagger b^{*\dagger}u^* = (vv^\dagger bu^\dagger)^*b^{*\dagger}u^*$, which by (10b) becomes

$$(bu^\dagger)^*b^{*\dagger}u^* = u^{*\dagger}b^*b^{*\dagger}u^* = u^{*\dagger}b^\dagger bu^* = \left[(ub^\dagger b)u^\dagger\right]^*.$$

But $ub^\dagger b = u$, and hence we arrive at $wx = (uu^\dagger)^* = uu^\dagger$.

Similarly $xw = v^\dagger b^{*\dagger}u^*u^{*\dagger}b^*v = v^\dagger b^{*\dagger}u^\dagger ub^*v$, which by (10a) reduces to

$$v^\dagger b^{*\dagger}b^*v = v^\dagger bb^\dagger v = v^\dagger v, \quad \text{since } bb^\dagger v = v.$$

Hence, $wxw = uu^\dagger w = w$ and $xwx = v^\dagger v(v^\dagger b^{*\dagger}u^*) = x$, as desired. Consequently, we may conclude that

$$(a + w)(a + w)^\dagger = aa^\dagger + uu^\dagger, (a + w)^\dagger(a + w) = a^\dagger a + v^\dagger v.$$

Finally let $c \geqslant a$, $c \geqslant b$, so that (5) holds. Then $(a + w)(a + w)^\dagger c = (aa^\dagger + uu^\dagger)c = a + uu^\dagger c = a + w$, since $uu^\dagger c = u^{*\dagger}u^*c = u^{*\dagger}b^*v$. Similarly $c(a + w)^\dagger(a + w) = c(a^\dagger a + v^\dagger v) = a + cv^\dagger v$ in which $cv^\dagger v = cv^*v^{*\dagger}$. Using (8) this equals $ub^*v^{*\dagger} = u^{*\dagger}(u^*ub^*)v^{*\dagger}$ and hence yields, with aid of (9),

$u^{*\dagger}(b^*vv^*)v^{*\dagger} = u^{*\dagger}b^*v = x$. Thus $a + w < c$ and consequently $a \vee b = a + u^{*\dagger}b^*v = a + ub^*v^{*\dagger}$.

3. Remarks and conclusions. Let us conclude this note with several remarks and conclusions.

(i) For projections (or Hermitian idempotents), e and f, the conditions (3) automatically hold because obviously $e(f - e)f = 0 = f(f - e)e$, $f(f - e) = f(1 - e)$, and $(f - e)f = (1 - e)f$. Thus

$$e \vee f = e + (1 - e)f[(1 - e)f]^\dagger = e + (1 - e)[(1 - e)f]^\dagger,$$

which is well known **[1]**, **[6]**.

(ii) When a and b *star-commute*, that is when a^*b and ba^* are Hermitian, then (3ii) and (3iii) hold *automatically*. To prove this we begin by observing that aa^* and bb^* commute. Since $(aa^*)^\dagger$ is the group inverse of aa^*, it follows by a result of Drazin **[4**, p. 208], that $(aa^*)^\dagger$ and bb^* also commute. Next, we note that

$$a^\dagger bb^* = a^*(aa^*)^\dagger bb^* = a^*bb^*(aa^*)^\dagger = b^*ba^*(aa^*)^\dagger = b^*ba^\dagger.$$

Lastly, we need the fact that $(b^*a)^\dagger = a^\dagger b^{*\dagger}$ and $(a^*b)^\dagger = b^\dagger a^{*\dagger}$, which may be verified directly or by using the reverse order law **[5**, p. 231]. Combining these see that $a^\dagger b = (a^\dagger bb^*)b^{*\dagger} = b^*ba^\dagger b^{*\dagger} = b^*b(b^*a)^\dagger = b^*b(a^*b)^\dagger = b^*bb^\dagger a^{*\dagger} = b^*a^{*\dagger}$, that is, $a^\dagger b$ is also Hermitian. Hence $aa^\dagger b = ab^*a^{*\dagger} = ba^*a^{*\dagger} = ba^\dagger a$, which implies that $u = v$. Thus, with aid of (11) $v^*b = u^*b = u^*u \in Ru$ while $bu^* = bv^* = vv^* \in vR$. This means that

$$a \vee b \text{ exists} \Leftrightarrow b(b^* - a^*)a = 0 = a(b^* - a^*)b. \tag{14}$$

In which case

$$a \vee b = a + u^{*\dagger}b^*v = a + u^{*\dagger}u^*u = a + (1 - aa^\dagger)b.$$

(iii) If a and b are *partial isometries*, such that $a^* = a^\dagger$ and $b^* = b^\dagger$, or equivalently $aa^*a = a$, $bb^*b = b$, then (14) *also* holds! The proof, however, is more delicate. First note that with aid of (3i) $u^*ub^* = b^*vv^*$. This allows us to conclude that bu^* and vb^* are both star-regular. Indeed,

$$(bu^*)(bu^*)^* = bu^*ub^* = bb^*vv^* = bb^*b(1 - a^\dagger a)v^* = vv^*,$$

and

$$(bu^*)^*(bu^*) = ub^*bu^* = (1 - aa^\dagger)bb^*bu^* = uu^*.$$

Similarly,

$$(v^*b)(v^*b)^* = v^*bb^*v = v^*bb^*b(1 - a^\dagger a) = v^*v$$

and

$$(v^*b)^*(v^*b) = b^*vv^*b = u^*ub^*b = u^*u,$$

all of which are regular by assumption. Hence

$$bu^* = (bu^*)(bu^*)^*(bu^*)^{*\dagger} = vv^*(bu^*)^{*\dagger} \in vR$$

and

$$v^*b = (v^*b)^{*\dagger}(v^*b)^*(v^*b) = (v^*b)^{*\dagger}u^*u \in Ru$$

as desired.

(iv) Using (1-21) of [5] we may rewrite (4) as

$$a \vee b = a + (1 - aa^\dagger)bb^*\big[(1 - a^\dagger a)b^*\big]^\dagger(1 - a^\dagger a),$$

however no $(a\text{–}b)$-symmetric formula is known at the present.

(v) Since $uu^\dagger c = u^{*\dagger}b^*v$ for all $c > a, b$, we have the following identity in $a \vee b - a$, $a \vee b - a = uu^\dagger(a \vee b - a)v^\dagger v$.

(vi) It is not known whether $a \vee b$ exists in a general star-regular ring, however it is anticipated that u and v will play a dominant role in its investigation.

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