PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 3, December 1979

SIGN COMPATIBLE EXPRESSIONS FOR MINORS OF THE MATRIX $I - A$

BY

D. J. HARTFIEL

ABSTRACT. Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix having row sums less than or equal to one. This paper shows that the ijth minor of $I - A$ can be expressed as

$$(-1)^{i+j} \sum \prod r_k a_{pq}$$

where

$$r_k = 1 - \sum_{s=1}^{n} a_{ks}$$

and each $\prod r_k a_{pq}$ is a product of exactly $n - 1$ numbers taken from r_k, a_{pq} for $k, p, q = 1, \ldots, n$. This theorem is then used to obtain perturbation results concerning the matrix $I - A$.

Perturbation results in matrix theory are concerned with estimating the error in matrix computations. This paper provides perturbation results for the matrix $I - A$ where $A = (a_{ij})$ is nonnegative having row sums less than or equal to one. The method by which these perturbation results are achieved is a variant of that given by Sengupta [2] in his work on comparing stochastic eigenvectors of two irreducible stochastic matrices. The method, as we apply it, first gives expressions for the minors of $I - A$, in terms of the entries of A, and then uses these expressions to produce useful perturbation results for this matrix.

The theorem of the paper produces expressions for the minors of $I - A$.

THEOREM. *Let A be an $n \times n$ nonnegative matrix having largest row sum less than or equal to one. Then*

$$|(I - A)_{ij}| = (-1)^{i+j} \sum \prod r_k a_{pq}$$

where

$$r_k = 1 - \sum_{s=1}^{n} a_{ks}$$

and each $\prod r_k a_{pq}$ is a product of exactly $n - 1$ numbers taken from r_k, a_{pq} for $k, p, q = 1, \ldots, n$.

Received by the editors January 16, 1978 and, in revised form, December 12, 1978.
AMS (MOS) subject classifications (1970). Primary 15A15; Secondary 15A09, 15A42.

PROOF. The method of proof is induction on n. The case $n = 2$ can be proved by checking all choices for i and j. Thus, suppose the theorem is true for all $n \times n$ matrices A, satisfying the hypothesis, where $n < n_1$. Now let A be an $n \times n$ matrix, satisfying the hypothesis, where $n = n_1$. The argument is divided into two cases. The first case to be considered is when $i \neq j$. Here we will assume $i < j$ as the case $i > j$ is argued similarly.

For this case, we first define an $n \times n$ matrix $E_{pq} = (e_{rs})$ where

$$e_{rs} = \begin{cases} 1, & \text{if } r = s \text{ and } r \neq p, r \neq q, \\ 1, & \text{if } r = p \text{ and } s = q, \\ 1, & \text{if } r = q \text{ and } s = p, \\ 0, & \text{otherwise.} \end{cases}$$

Let $P = E_{12}E_{23} \cdots E_{i-1,i}$, a permutation matrix, and set $P(I - A)P^t = I - PAP^t = I - B$. Then

$$|(I - A)_{ij}| = (-1)^{i-1}|(I - B)_{1j}|.$$

Hence we need only prove the result for $i = 1$ and $j > 1$.

For this then we expand $|(I - A)_{1j}|$ about the 1st column achieving that

$$|(I - A)_{1j}| = \sum_{s>1} (-1)^{s-1+1}(-a_{s1})|[(I - A)_{1j}]_{s1}|$$

where $[(I - A)_{1j}]_{s1}$ denotes the matrix obtained from $(I - A)$ by deleting rows $1, s$ and columns j, 1.

Now noting that $[(I - A)_{1j}]_{s1} = [(I - A)_{11}]_{sj}$ and applying the induction hypothesis yields that

$$\begin{aligned} |(I - A)_{1j}| &= \sum_{s>1} (-1)^{s+1} a_{s1}|[(I - A)_{11}]_{sj}| \\ &= \sum_{s>1} (-1)^{s+1} a_{s1}\Big((-1)^{s+j} \sum \prod (r_k + a_{k1})a_{pq}\Big) \\ &= (-1)^{1+j} \sum \prod r_k a_{pq} \end{aligned}$$

where each $\prod r_k a_{pq}$ is a product of exactly $n - 1$ numbers taken from r_k, a_{pq} for $k, p, q = 1, \ldots, n$.

For the case $i = j$, we assume without loss of generality that $i = j = 1$. Write

$$(I - A)_{11} = \begin{bmatrix} r_2 + \sum_{s \neq 2} a_{2s} & -a_{23} & \cdots & -a_{2n} \\ -a_{32} & r_3 + \sum_{s \neq 3} a_{3s} & \cdots & -a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n2} & -a_{n3} & \cdots & r_n + \sum_{s \neq n} a_{ns} \end{bmatrix}.$$

Adding columns two through $n-1$ to column one yields

$$B=\begin{bmatrix} r_2+a_{21} & -a_{23} & \cdots & -a_{2n} \\ r_3+a_{31} & r_3+\sum\limits_{s\neq 3} a_{3s} & \cdots & -a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ r_n+a_{n1} & -a_{n3} & \cdots & r_n+\sum\limits_{s\neq n} a_{ns} \end{bmatrix}.$$

Expanding the determinant about the first column of B and noting that $B_{s1}=[(I-A)_{11}]_{s+1,2}$ yields, by the induction hypothesis, that

$$\begin{aligned}|(I-A)_{11}| &= \sum_{s=1}^{n-1}(-1)^{s+1}(r_{s+1}+a_{s+1,1})|[(I-A)_{11}]_{s+1,2}| \\ &= \sum_{s=1}^{n-1}(-1)^{s+1}(r_{s+1}+a_{s+1,1})\left((-1)^{s+1}\left(\sum\prod(r_k+a_{k1})a_{pq}\right)\right. \\ &= \sum\prod r_k a_{pq}\end{aligned}$$

where each $\prod r_k a_{pq}$ is a product of exactly $n-1$ numbers taken from r_k, a_{pq} for $k,p,q=1,\ldots,n$. $\square$

This theorem is now applied to yield our first perturbation result. This result estimates the error in computing $(I-A)^{-1}$, when it exists.

COROLLARY 1. *Let A and $\hat{A}$ be $n\times n$ nonnegative matrices with row sums less than or equal to one and having spectral radius less than* 1. *Set $B=(I-A)^{-1}$ and $\hat{B}=(I-\hat{A})^{-1}$. If*

(1) $a_{ij}\leq\hat{\theta}\hat{a}_{ij}$, $r_k\leq\hat{\theta}\hat{r}_k$ and

(2) $\hat{a}_{ij}\leq\theta a_{ij}$, $\hat{r}_k\leq\theta r_k$

then, for all $b_{ij}\neq 0$, $(\hat{b}_{ij}-b_{ij})/b_{ij}\leq\theta^{n-1}\hat{\theta}^n-1$.

PROOF. Note that

$$b_{ij}=\frac{(-1)^{i+j}|(I-A)_{ji}|}{|(I-A)|}\quad\text{and}\quad\hat{b}_{ij}=\frac{(-1)^{i+j}|(I-\hat{A})_{ji}|}{|(I-\hat{A})|}.$$

Thus, if $b_{ij}\neq 0$, application of the theorem yields that

$$\frac{\hat{b}_{ij}}{b_{ij}}=\frac{|(I-\hat{A})_{ji}|\,|I-A|}{|(I-A)_{ji}|\,|I-\hat{A}|}\leq\frac{\theta^{n-1}|(I-A)_{ji}|\hat{\theta}^n|I-\hat{A}|}{|(I-A)_{ji}|\,|I-\hat{A}|}\leq\theta^{n-1}\hat{\theta}^n.$$

Hence,

$$\frac{\hat{b}_{ij}-b_{ij}}{b_{ij}}\leq\theta^{n-1}\hat{\theta}^n-1.\quad\square$$

A second perturbation result estimates the error in solving Leontif's open economic model.

COROLLARY 2. *Let A and $\hat{A}$ be $n \times n$ nonnegative matrices having largest row sums less than or equal to one and having spectral radii less than one. Let b and $\hat{b}$ be $1 \times n$ nonnegative vectors with $x(I - A) = b$ and $\hat{x}(I - \hat{A}) = \hat{b}$. If*

(1) $a_{ij} \leqslant \hat{\theta}\hat{a}_{ij}$, $r_k(A) \leqslant \hat{\theta} r_k(\hat{A})$, $b_i \leqslant \hat{\theta}\hat{b}_i$ *and*

(2) $\hat{a}_{ij} < \theta a_{ij}$, $r_k(\hat{A}) \leqslant \theta r_k(A)$, $\hat{b}_i \leqslant \theta b_i$

then, for all $x_i \neq 0$, $(\hat{x}_i - x_i)/x_i \leqslant (\theta\hat{\theta})^n - 1$.

PROOF. Note first that $x = (I - A)^{-1}b$ and $\hat{x} = (I - \hat{A})^{-1}\hat{b}$. Then, if $x_i \neq 0$, application of the theorem yields that

$$\frac{\hat{x}_i}{x_i} = \frac{\Sigma_r(-1)^{i+r}|(I - \hat{A})_{ri}|\hat{b}_r|I - A|}{|I - \hat{A}|\Sigma_r(-1)^{i+r}|(I - A)_{ri}|b_r} \leqslant (\theta\hat{\theta})^n.$$

Hence

$$\frac{\hat{x}_i - x_i}{x_i} \leqslant (\theta\hat{\theta})^n - 1. \quad \square$$

The last perturbation result estimates the error in computing stochastic eigenvectors for stochastic matrices.

COROLLARY 3. *Let A and $\hat{A}$ be $n \times n$ irreducible stochastic matrices. Suppose α and $\hat{\alpha}$ are stochastic eigenvectors, belonging to one, for A and $\hat{A}$ respectively. If* (1) $a_{ij} \leqslant \hat{\theta}\hat{a}_{ij}$ *and* (2) $\hat{a}_{ij} \leqslant \theta a_{ij}$ *then* $(\hat{\alpha}_i - \alpha_i)/\alpha_i < (\theta\hat{\theta})^{n-1} - 1$.

PROOF. First note that the Perron-Frobenius theory [1] gives that if α and $\hat{\alpha}$ are stochastic eigenvectors, belonging to one, for A and $\hat{A}$ respectively, then α and $\hat{\alpha}$ are the unique solutions to

$$\alpha(I - A) = 0 \text{ with } \sum \alpha_i = 1 \quad \text{and} \quad \hat{\alpha}(I - \hat{A}) = 0 \text{ with } \sum \hat{\alpha}_i = 1.$$

Further, by the Perron-Frobenius theory, rank $(I - A) = \text{rank}(I - \hat{A}) = n - 1$ with the first $n - 1$ columns of both $I - A$ and $I - \hat{A}$ being linearly independent. Hence, the above equations are equivalent to

$$\alpha[(I - A)_n e] = e_n \quad \text{and} \quad \hat{\alpha}[(I - \hat{A})_n e] = e_n$$

where $(I - A)_n$ and $(I - \hat{A})_n$ are obtained by deleting the nth column of $(I - A)$ and $(I - \hat{A})$ respectively. Further e_i is the (0, 1)-vector having its only nonzero entry in the ith position and $e = e_1 + \cdots + e_n$. Now, by Cramer's rule

$$\alpha_i = (-1)^{i+n}\det([(I - A)_n e]_{in})/\det[(I - A)_n e]$$

and

$$\hat{\alpha}_i = (-1)^{i+n}\det([(I - \hat{A})_n e]_{in})/\det[(I - \hat{A})_n e].$$

Thus

$$\frac{\hat{\alpha}_i}{\alpha_i} = \frac{\det([(I - \hat{A})_n e]_{in})\det[(I - A)_n e]}{\det([(I - A)_n e]_{in})\det[(I - \hat{A})_n e]}.$$

Noting that $\det[(I-\hat{A})_n e]_{in} = \det[(I-\hat{A})_{in}]$ and $\det[(I-A)_n e]_{in} = \det[(I-A)_{in}]$ and expanding $\det[(I-A)_n e]$ and $\det[(I-\hat{A})_n e]$ about the last column yields, by applying the theorem, that

$$\frac{\hat{\alpha}_i}{\alpha_i} < \frac{\left(\theta^{n-1}\sum\prod r_k a_{pq}\right)\left(\hat{\theta}^{n-1}\sum\prod \hat{r}_k \hat{a}_{pq}\right)}{\left(\sum\prod r_k a_{pq}\right)\left(\sum\prod \hat{r}_k \hat{a}_{pq}\right)} < (\theta\hat{\theta})^{n-1}.$$

Hence,

$$\frac{\hat{\alpha}_i - \alpha_i}{\alpha_i} < (\theta\hat{\theta})^{n-1} - 1. \quad \square$$

References

1. F. R. Gantmacher, *The theory of matrices*. Vol. 2, Chelsea, New York, 1960.
2. Sailes Kumar Sengupta, *Comparison of eigenvectors of irreducible stochastic matrices*, Linear Algebra and Appl. **12** (1975), 101–110.

Department of Mathematics, Texas A&M University, College Station, Texas 77843