PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 3, December 1979

ON GENERATORS OF IDEALS

T. T. MOH[1]

ABSTRACT. In our previous work we had constructed some space curves to illustrate the unboundedness of generators of prime ideals in analytic geometry. In this article we further study these space curves to establish the exact number of minimal generators for the corresponding ideals and to express them as determinant ideals.

In our previous work [2] we had constructed a set $\{C_n\}$ of rational irreducible curves in 3-dimensional affine space which are analytically irreducible at the origin. Moreover the corresponding prime ideal ideals P_n need at least n generators in $k[[x, y, z]]$ and hence in $k[x, y, z]$ where k is a field of characteristic zero. The purpose of those constructions are to establish the unboundedness of generators of prime ideals in analytic geometry. For a background discussion about the set of prime ideals $\{P_n\}$ the reader is referred to Sally's book [3]. In fact Sally finds an interesting argument to drop the restriction on the characteristic of the ground field k.

The purpose of this article is to establish that the said prime ideals $\{P_n\}$ require precisely $n + 1$ elements as the minimal number of generators. Moreover we should indicate a constructive way to find minimal generators. One interesting question is to determine if C_n is locally a set-theoretic complete intersection. With a suggestion of M. Hochster we shall indicate a way to express P_n as a determinant ideal.

1. Definitions. We shall use the notations of [2]. Let n be an odd positive integer and $m = (n + 1)/2$. Let S be the semigroup generated by $(n + 1)$ and $(n + 2)$. Let λ be an integer $> n(n + 1)m$ with $(\lambda, m) = 1$. Let p be a mapping: $k[[x, y, z]] \to k[[t]]$ defined by

$$\rho(x) = t^{nm} + t^{nm+\lambda}, \qquad \rho(y) = t^{(n+1)m},$$
$$\rho(z) = t^{(n+2)m}$$

where $k[[x, y, z]]$, $k[[t]]$ are power series rings in symbols x, y, z, t. Let $P_n = P = \ker \rho$.

Let σ-weight be given by

$$\sigma(x) = x^n, \qquad \sigma(y) = y^{n+1}, \qquad \sigma(z) = z^{n+2}.$$

Received by the editors November 13, 1978 and, in revised form, December 28, 1978.

AMS (MOS) subject classifications (1970). Primary 13A15, 14H45.

Key words and phrases. Generators of ideals, determinant ideal, power series ring, binomial vectors, σ-leading form.

[1]This work has been supported by N.S.F. at Purdue University.

Let $W_r = \{\sigma\text{-homogeneous forms of } \sigma\text{-order } r\} \cup 0$. Let $d_r = \dim W_r$. Let $V_r = W_r \cap \{\sigma\text{-leading forms of elements in } P\} \cup 0$. Let $c_r = \dim V_r$.

Let $D = (d_{ij})$ be the $n \times (n+1)$ matrix with $d_{ij} = (i-j)(n+1) + j(n+2)$. We observe $d_{ij} \in S \Leftrightarrow i \geqslant j$.

For the definitions of binomial vectors and the mapping b_m the reader is referred to Definitions 2.1 and 4.2 of [2].

2. The minimal generators of P_n. Let us recall the following obvious lemma.

LEMMA. *Let $f_1, \ldots, f_s$ be elements in P_n with σ-leading forms generating the σ-leading ideal of P_n. Then $(f_1, \ldots, f_s) = P_n$.*

THEOREM. *The prime ideal P_n needs at least $n+1$ generators. There are $f_1, \ldots, f_{n+1} \in P_n$ such that:*

(1) *the σ-leading form of $f_i \in V_{n^2+n+i-1}$ for $i = 1, \ldots, n$;*

(2) *x(the σ-leading form of f_1) and the σ-leading form of f_{n+1} generate V_{n^2+2n}. Moreover any $f_1, \ldots, f_{n+1}$ satisfying conditions* (1) *and* (2) *generate P_n.*

PROOF. Let $\{g_1, \ldots, g_s\}$ be a set of generators for P_n. The proof of Theorem 4.3 of [2] shows that $\{g_1, \ldots, g_s\}$ may be replaced by $\{f_1, \ldots, f_n, g_{n+1}, \ldots, g_s\}$ such that

$$\sigma\text{-ord} f_i = n^2 + n + (i-1), \qquad \forall i = 1, \ldots, n,$$

and

$$\sigma\text{-ord}\, g_j \geqslant n^2 + 2n, \qquad \forall j = n+1, \ldots, s.$$

Note that it follows from Theorems 4.1 and 4.2 of [2] that $c_{n^2+2n} = 2$. Since $f(x, y, z) f_i$ has a σ-order $n^2 + 2n$ only if σ-ord $f = n$ and σ-ord $f_i = n^2 + n$. Namely $f = ax + \ldots, f_i = f_1$. Hence we need at least one more element f_{n+1} of σ-order $n^2 + 2n$ to generate P_n. Thus P_n requires at least $n+1$ generators. Moreover we have established the existence of elements $f_1, \ldots, f_{n+1}$ which satisfy the conditions (1) and (2).

It follows from the lemma that it suffices to prove the σ-leading forms of $f_1, \ldots, f_{n+1}$ generate the σ-leading ideal of P_n. Let us consider V_r for $r > n^2 + 2n$. Let us consider two cases, (1) $n^2 + 2n < r < (n+1)(n+2)$ and (2) $(n+1)(n+2) \leqslant r$.

Case 1. Let $r = n^2 + 2n + i$ for $1 \leqslant i \leqslant n+1$. Let $d_{i_1 j_1}, \ldots, d_{i_m j_m}$ be elements in S with residue $(r-1) \bmod n$ (cf. Theorem 3.1 of [2]). Let

$$d_{i_1 j_1} = \beta_1(n+1) + r_1(n+2), \ldots, d_{i_m j_m} = \beta_m(n+1) + r_m(n+2).$$

Note that β's, r's are all nonnegative. Let $x^{\alpha_1} y^{\beta_1+1} z^{r_1}, \ldots, x^{\alpha_m} y^{\beta_m+1} z^{r_m}$ be the corresponding monomials with σ-order r. Note that $r > d_{i,j} + n + 1$, hence $\alpha_i > 0$. Moreover all α_i's are distinct. Recall the definition of b_m (cf. Definition 4.2 of [2]). It follows from Theorem 2.1 of [2] that $b_m(x^{\alpha_1} y^{\beta_1+1} z^{r_1}), \ldots, b_m(x^{\alpha_m} y^{\beta_m+1} z^{r_m})$ form a basis for the image of b_m: $W_r \to k^m$. Let $x^{\alpha} y^{\beta} z^{r} \in W_r$. Note that either $\alpha > 0$ or $\beta > 0$. Suppose

$$a_\alpha x^\alpha y^\beta z^r + \sum_{i=1}^{m} a_{\alpha_i} x^{\alpha_i} y^{\beta_i+1} z^{r_i} \in V_r = \ker b_m.$$

If $\alpha > 0$ then the whole relation is divisible by x, i.e. the relation comes from V_{r-n} by multiplying x. If $\beta > 0$ then the whole relation is divisible by y, i.e. the relation comes from V_{r-n-1} by multiplying y.

Case 2. Let $(n+1)(n+2) \leqslant r$. Let $d_{i_1 j_1}, \ldots, d_{i_m j_m}$ be elements in S with residue $(r-3) \bmod n$. Let

$$d_{i_1 j_1} = \beta_1(n+1) + r_1(n+2), \ldots, d_{i_m j_m} = \beta_m(n+1) + r_m(n+2).$$

Let $x^{\alpha_1} y^{\beta_1+1} z^{r_1+1}, \ldots, x^{\alpha_m} y^{\beta_m+1} z^{r_m+1}$ be the corresponding monomials with σ-order r. Note that $r > d_{ij} + 2n + 3$. Hence all α_i's are positive and distinct. Let $x^\alpha y^\beta z^r \in W_r$. Then at least one of α, β and r is positive. Thus there is a relation $a_\alpha x^\alpha y^\beta z^r + \sum a_{\alpha_i} x^{\alpha_i} y^{\beta_i+1} z^{r_i+1}$ which comes from a relation which comes from V_{r-n} or V_{r-n-1} or V_{r-n-2}. Q.E.D.

3. Determinant ideals. One of the interesting ways of expressing the ideal P_n is to express it as the ideal generated by $(n \times n)$ subdeterminants of an $n \times (n+1)$ matrix. To this purpose it suffices to write down certain systems of equations satisfied by a set of generators of $n+1$ elements. Recall the following lemma:

LEMMA. *We have* $C_r = 2$ *for* $r = n^2 + 2n + 2, \ldots, n^2 + 3n - 1$ *and* $C_r = 3$ *for* $r = n^2 + 3n, n^2 + 3n + 1$.

PROOF. Statement (3) of Theorem 4.1 and Statement (2) of Theorem 4.2 of [2].

Let $f_1, \ldots, f_{n+1}$ be a set of generators for P_n as specified by our preceding theorem. Then $zf_1, yf_2, xf_3, 0f_4, \ldots, 0f_{n+1}$ are element of σ-order $n^2 + 2n + 2$. According to the preceding lemma the σ-leading forms of zf_1, yf_2, xf_3 satisfy a nontrivial linear relation. Hence $zf_1, yf_2, xf_3, f_4, \ldots, f_{n+1}$ satisfy an equation

$$a_{11} z f_1 + a_{12} y f_2 + a_{13} x f_3 + a_{14} f_4 + \cdots + a_{1n+1} f_{n+1} = 0$$

where $a_{1i}(0,0,0) = 0, \forall i \geqslant 4$, and

$$\begin{aligned} a_{11}(0,0,0)z(&\sigma\text{-leading form of } f_1) \\ &+ a_{12}(0,0,0)y(\sigma\text{-leading form of } f_2) \\ &+ a_{13}(0,0,0)x(\sigma\text{-leading form of } f_3) = 0 \end{aligned}$$

is a nontrivial relation.

Let $1 \leqslant j \leqslant n-2$. Then $0f_1, 0f_2, \ldots, 0f_{j-1}, zf_j, yf_{j+1}, xf_{j+2}, 0f_{j+3}, \ldots, 0f_{n+1}$ are elements of σ-order $n^2 + 2n + j + 1$. According to the preceding lemma the σ-leading forms of zf_j, yf_{j+1}, xf_{j+2} satisfy a nontrivial linear relation. Hence $f_1, f_2, \ldots, zf_j, yf_{j+1}, xf_{j+2}, f_{j+3}, \ldots, f_{n+1}$ satisfy an equation

$$\sum_{i=1}^{j-1} a_{ji} f_i + a_{jj} z f_j + a_{jj+1} y f_{j+1} + a_{jj+2} x f_{j+2} + \sum_{i=j+3}^{n+1} a_{ji} f_i = 0$$

where $a_{ji}(0, 0, 0) = 0, \forall i \neq j, j+1, j+2$ and

$$\begin{aligned} a_{jj}(0,0,0)z(\sigma\text{-leading form of } f_j) \\ + a_{jj+1}(0,0,0)y(\sigma\text{-leading form of } f_{j+1}) \\ + a_{jj+2}(0,0,0)x(\sigma\text{-leading form of } f_{j+2}) = 0 \end{aligned}$$

is a nontrivial linear relation.

For $j = n-1$ (resp. $j = n$), we shall consider $x^2 f_1$, zf_{n-1}, yf_n, xf_{n+1} (resp. xyf_1, x^2f_2, zf_n, yf_{n+1}). Then they are elements of σ-order $n^2 + 3n$ (resp. $n^2 + 3n + 1$). Similarly we get two equations. It follows from linear algebra that

$$(f_1 : f_2 : f_3 : \cdots : f_{n+1}) \sim (\Delta_1 : \Delta_2 : \Delta_3 : \cdots : \Delta_{n+1})$$

where Δ_i is a subdeterminant of the $n \times (n+1)$ matrix Δ of coefficients. It remains to show that at least one $\Delta_i \neq 0$ and that σ-order of Δ_i = σ-order of f_i.

Let $\bar{\Delta}$ equal the corresponding subdeterminant after substituting a_{ji} by $a_{ji}(0, 0, 0)$. Then it suffices to show $\bar{\Delta}_1 \neq 0$ and σ-order of $\bar{\Delta}_1 = n^2 + n$. We have

$$\bar{\Delta} = \begin{bmatrix} {}^*z & {}^*y, & {}^*x, & 0, & & & 0 \\ 0 & {}^*z, & {}^*y, & {}^*x, & 0 & & 0 \\ 0 & 0 & {}^*z, & {}^*y, & {}^*x, & 0 & 0 \\ & 0 & & \ddots & & \ddots & \\ & & & & \ddots & \ddots & 0 \\ {}^*x^2 & 0 & & & & {}^*y, & {}^*x \\ {}^*xy & {}^*x^2, & 0 & & & {}^*z, & {}^*y \end{bmatrix}$$

where *'s are constant and $\bar{\Delta}$ gives the relations among the σ-leading forms of $f_1, \ldots, f_{n+1}$. Let us show the coefficients of y's are never zeroes. Note that none of the σ-leading forms of $f_1, \ldots, f_{n+1}$ is divisible by x, y or z. Hence if one of the coefficients of y is zero, then the coefficient of z in the same row must be zero. We easily get a contradiction. Now it is clear that $\bar{\Delta}_1 \neq 0$ and σ-order of $\bar{\Delta}_1 = n^2 + n$. Hence $(f_1, \ldots, f_{n+1}) = (\Delta_1, \Delta_2, \ldots, \Delta_{n+1})$. Thus we have established:

Theorem. *The ideal P_n is an ideal generated by $n \times n$ subdeterminants of an $n \times (n+1)$ matrix with entries in $k[[x, y, z]]$.*

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Department of Mathematics, Purdue University, West Lafayette, Indiana 47907