ON GENERATORS OF IDEALS

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Abstract. In our previous work we had constructed some space curves to illustrate the unboundedness of generators of prime ideals in analytic geometry. In this article we further study these space curves to establish the exact number of minimal generators for the corresponding ideals and to express them as determinant ideals.

In our previous work [2] we had constructed a set \( \{C_n\} \) of rational irreducible curves in 3-dimensional affine space which are analytically irreducible at the origin. Moreover the corresponding prime ideal ideals \( P_n \) need at least \( n \) generators in \( k[[x, y, z]] \) and hence in \( k[x, y, z] \) where \( k \) is a field of characteristic zero. The purpose of those constructions are to establish the unboundedness of generators of prime ideals in analytic geometry. For a background discussion about the set of prime ideals \( \{P_n\} \) the reader is referred to Sally's book [3]. In fact Sally finds an interesting argument to drop the restriction on the characteristic of the ground field \( k \).

The purpose of this article is to establish that the said prime ideals \( \{P_n\} \) require precisely \( n + 1 \) elements as the minimal number of generators. Moreover we should indicate a constructive way to find minimal generators. One interesting question is to determine if \( C_n \) is locally a set-theoretic complete intersection. With a suggestion of M. Hochster we shall indicate a way to express \( P_n \) as a determinant ideal.

1. Definitions. We shall use the notations of [2]. Let \( n \) be an odd positive integer and \( m = (n + 1)/2 \). Let \( S \) be the semigroup generated by \( n + 1 \) and \( n + 2 \). Let \( \lambda \) be an integer \( > n(n + 1)m \) with \( (\lambda, m) = 1 \). Let \( p \) be a mapping: \( k[[x, y, z]] \rightarrow k[[t]] \) defined by

\[
p(x) = t^{nm} + t^{nm+\lambda}, \quad p(y) = t^{(n+1)m}, \quad p(z) = t^{(n+2)m}
\]

where \( k[[x, y, z]], k[[t]] \) are power series rings in symbols \( x, y, z, t \). Let \( P_n = P = \ker p \).

Let \( \sigma \)-weight be given by

\[
\sigma(x) = x^n, \quad \sigma(y) = y^{n+1}, \quad \sigma(z) = z^{n+2}.
\]
Let $W_r = \{\sigma \text{-homogeneous forms of } \sigma \text{-order } r\} \cup \emptyset$. Let $d_r = \dim W_r$. Let $V_r = W_r \cap \{\sigma \text{-leading forms of elements in } P\} \cup \emptyset$. Let $c_r = \dim V_r$.

Let $D = (d_{ij})$ be the $n \times (n + 1)$ matrix with $d_{ij} = (i - j)(n + 1) + j(n + 2)$. We observe $d_{ij} \in S \iff i \geq j$.

For the definitions of binomial vectors and the mapping $b_m$ the reader is referred to Definitions 2.1 and 4.2 of [2].

2. The minimal generators of $P_n$. Let us recall the following obvious lemma.

**Lemma.** Let $f_1, \ldots, f_s$ be elements in $P_n$ with $\sigma$-leading forms generating the $\sigma$-leading ideal of $P_n$. Then $(f_1, \ldots, f_s) = P_n$.

**Theorem.** The prime ideal $P_n$ needs at least $n + 1$ generators. There are $f_1, \ldots, f_{n+1} \in P_n$ such that:

1. the $\sigma$-leading form of $f_i$ lies in $V_{n^2 + n + i - 1}$ for $i = 1, \ldots, n$;
2. $x$ (the $\sigma$-leading form of $f_i$) and the $\sigma$-leading form of $f_{n+1}$ generate $V_{n^2 + 2n}$.

Moreover any $f_1, \ldots, f_{n+1}$ satisfying conditions (1) and (2) generate $P_n$.

**Proof.** Let $(g_1, \ldots, g_s)$ be a set of generators for $P_n$. The proof of Theorem 4.3 of [2] shows that $(g_1, \ldots, g_s)$ may be replaced by $(f_1, \ldots, f_n, g_{n+1}, \ldots, g_s)$ such that

$$\sigma \text{-ord } f_i = n^2 + n + (i - 1), \quad \forall i = 1, \ldots, n,$$

and

$$\sigma \text{-ord } g_j \geq n^2 + 2n, \quad \forall j = n + 1, \ldots, s.$$

Note that it follows from Theorems 4.1 and 4.2 of [2] that $c_{n^2 + 2n} = 2$. Since $f(x, y, z) f_i$ has a $\sigma$-order $n^2 + 2n$ only if $\sigma$-ord $f = n$ and $\sigma$-ord $f_i = n^2 + n$. Namely $f = ax + \ldots, f_i = f_i$. Hence we need at least one more element $f_{n+1}$ of $\sigma$-order $n^2 + 2n$ to generate $P_n$. Thus $P_n$ requires at least $n + 1$ generators. Moreover we have established the existence of elements $f_1, \ldots, f_{n+1}$ which satisfy the conditions (1) and (2).

It follows from the lemma that it suffices to prove the $\sigma$-leading forms of $f_1, \ldots, f_{n+1}$ generate the $\sigma$-leading ideal of $P_n$. Let us consider $V_r$ for $r > n^2 + 2n$. Let us consider two cases, (1) $n^2 + 2n < r < (n + 1)(n + 2)$ and (2) $(n + 1)(n + 2) < r$.

**Case 1.** Let $r = n^2 + 2n + i$ for $1 < i < n + 1$. Let $d_{i,1}, \ldots, d_{i,n}$ be elements in $S$ with residue $(r - 1) \mod n$ (cf. Theorem 3.1 of [2]). Let

$$d_{i,1} = \beta_1(n + 1) + r_i(n + 2), \ldots, d_{i,n} = \beta_m(n + 1) + r_m(n + 2).$$

Note that $\beta_i$'s, $r_i$'s are all nonnegative. Let $x^{a_1} y^{\beta_1 + 1} z^r_1, \ldots, x^{a_m} y^{\beta_m + 1} z^r_m$ be the corresponding monomials with $\sigma$-order $r$. Note that $r > d_{i,1} + n + 1$, hence $\alpha_i > 0$. Moreover all $\alpha_i$'s are distinct. Recall the definition of $b_m$ (cf. Definition 4.2 of [2]). It follows from Theorem 2.1 of [2] that $b_m(x^{a_1} y^{\beta_1 + 1} z^r_1), \ldots, b_m(x^{a_m} y^{\beta_m + 1} z^r_m)$ form a basis for the image of $b_m$: $W_r \to k^m$. Let $x^\alpha y^{\beta} z^r \in W_r$. Note that either $\alpha > 0$ or $\beta > 0$. Suppose
If $a > 0$ then the whole relation is divisible by $x$, i.e. the relation comes from $V_{r-n}$ by multiplying $x$. If $\beta > 0$ then the whole relation is divisible by $y$, i.e. the relation comes from $V_{r-n-1}$ by multiplying $y$.

**Case 2.** Let $(n+1)(n+2) < r$. Let $d_{i_1}, \ldots, d_{i_m}$ be elements in $S$ with residue $(r-3) \mod n$. Let

$$d_{i_1} = \beta_1(n+1) + r_1(n+2), \ldots, d_{i_m} = \beta_m(n+1) + r_m(n+2).$$

Let $x^{a_1^{\alpha_1} \beta_1^{\beta_1} + 1}r_1^{n+1}, \ldots, x^{a_m^{\alpha_m} \beta_m^{\beta_m} + 1}r_m^{n+1}$ be the corresponding monomials with $\sigma$-order $r$. Note that $r > d_{i_j} + 2n + 3$. Hence all $\alpha_i$'s are positive and distinct. Let $x^{a_1^{\alpha_1} \beta_1^{\beta_1} + 1}r_1^{n+1}$ be the corresponding monomials with $\sigma$-order $r$. Then at least one of $a, \beta$ and $r$ is positive. Thus there is a relation $a_1x^{a_1^{\alpha_1} \beta_1^{\beta_1} + 1}r_1^{n+1} + \sum a_i x_i^{a_i^{\alpha_i} \beta_i^{\beta_i} + 1}r_i^{n+1}$ which comes from a relation which comes from $V_{r-n}$ or $V_{r-n-1}$ or $V_{r-n-2}$. Q.E.D.

**3. Determinant ideals.** One of the interesting ways of expressing the ideal $P_n$ is to express it as the ideal generated by $(n \times n)$ subdeterminants of an $n \times (n+1)$ matrix. To this purpose it suffices to write down certain systems of equations satisfied by a set of generators of $n+1$ elements. Recall the following lemma:

**Lemma.** We have $C_r = 2$ for $r = n^2 + 2n + 2, \ldots, n^2 + 3n - 1$ and $C_r = 3$ for $r = n^2 + 3n, n^2 + 3n + 1$.

**Proof.** Statement (3) of Theorem 4.1 and Statement (2) of Theorem 4.2 of [2].

Let $f_1, \ldots, f_{n+1}$ be a set of generators for $P_n$ as specified by our preceding theorem. Then $zf_1, yf_2, xf_3, 0f_4, \ldots, 0f_{n+1}$ are element of $\sigma$-order $n^2 + 2n + 2$. According to the preceding lemma the $\sigma$-leading forms of $zf_1, yf_2, xf_3$ satisfy a nontrivial linear relation. Hence $zf_1, yf_2, xf_3, f_4, \ldots, f_{n+1}$ satisfy an equation

$$a_{11}zf_1 + a_{12}yf_2 + a_{13}xf_3 + a_{14}f_4 + \cdots + a_{1n+1}f_{n+1} = 0$$

where $a_{1i}(0, 0, 0) = 0, \forall i > 4$, and

$$a_{11}(0, 0, 0)z(\sigma\text{-leading form of } f_i) + a_{12}(0, 0, 0)y(\sigma\text{-leading form of } f_2) + a_{13}(0, 0, 0)x(\sigma\text{-leading form of } f_3) = 0$$

is a nontrivial relation.

Let $1 < j < n - 2$. Then $0f_1, 0f_2, \ldots, 0f_{j-1}, zf_j, yf_{j+1}, xf_{j+2}, 0f_{j+3}, \ldots, 0f_{n+1}$ are elements of $\sigma$-order $n^2 + 2n + j + 1$. According to the preceding lemma the $\sigma$-leading forms of $zf_j, yf_{j+1}, xf_{j+2}$ satisfy a nontrivial linear relation. Hence $f_1, f_2, \ldots, zf_j, yf_{j+1}, xf_{j+2}, f_{j+3}, \ldots, f_{n+1}$ satisfy an equation

$$\sum_{i=1}^{j-1} a_if_i + a_jzf_j + a_{j+1}yf_{j+1} + a_{j+2}xf_{j+2} + \sum_{i=j+3}^{n+1} a_if_i = 0$$
where $a_i(0, 0, 0) = 0, \forall i \neq j, j + 1, j + 2$ and

$$a_{j+1}(0, 0, 0)z (\sigma\text{-leading form of } f_j)$$

$$+ a_{j+1}(0, 0, 0)y (\sigma\text{-leading form of } f_{j+1})$$

$$+ a_{j+2}(0, 0, 0)x (\sigma\text{-leading form of } f_{j+2}) = 0$$

is a nontrivial linear relation.

For $j = n - 1$ (resp. $j = n$), we shall consider $x^2 f_1, z f_{n-1}, y f_n, x f_{n+1}$ (resp. $x y f_1, x^2 f_2, z f_n, y f_{n+1}$). Then they are elements of $\sigma$-order $n^2 + 3n$ (resp. $n^2 + 3n + 1$). Similarly we get two equations. It follows from linear algebra that

$$(f_1 : f_2 : \cdots : f_{n+1}) \sim (\Delta_1 : \Delta_2 : \cdots : \Delta_{n+1})$$

where $\Delta_i$ is a subdeterminant of the $n \times (n + 1)$ matrix $\Delta$ of coefficients. It remains to show that at least one $\Delta_i \neq 0$ and that $\sigma$-order of $\Delta_i = \sigma$-order of $f_i$.

Let $\bar{\Delta}$ equal the corresponding subdeterminant after substituting $a_{ij}$ by $a_{ij}(0, 0, 0)$. Then it suffices to show $\bar{\Delta}_1 \neq 0$ and $\sigma$-order of $\bar{\Delta}_1 = n^2 + n$. We have

$$\bar{\Delta} = \begin{bmatrix}
  *z & *y, & *x, & 0, & 0
  0 & *z, & *y, & *x, & 0
  0 & 0 & *z, & *y, & *x, & 0
  0 & 0 & 0 & \cdots & \cdots & 0
  *x^2 & 0 & \cdots & *y, & *x
  *xy & *x^2 & 0 & *z, & *y
\end{bmatrix}$$

where **'s are constant and $\bar{\Delta}$ gives the relations among the $\sigma$-leading forms of $f_1, \ldots, f_{n+1}$. Let us show the coefficients of $y$'s are never zeroes. Note that none of the $\sigma$-leading forms of $f_1, \ldots, f_{n+1}$ is divisible by $x, y$ or $z$. Hence if one of the coefficients of $y$ is zero, then the coefficient of $z$ in the same row must be zero. We easily get a contradiction. Now it is clear that $\bar{\Delta}_1 \neq 0$ and $\sigma$-order of $\bar{\Delta}_1 = n^2 + n$. Hence $(f_1, \ldots, f_{n+1}) = (\Delta_1, \Delta_2, \ldots, \Delta_{n+1})$. Thus we have established:

**Theorem.** The ideal $P_n$ is an ideal generated by $n \times n$ subdeterminants of an $n \times (n + 1)$ matrix with entries in $k[[x, y, z]]$.

**Bibliography**


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