SOME DECIDABLE DIOPHANTINE PROBLEMS:

POSITIVE SOLUTION TO A PROBLEM OF

DAVIS, MATIJASEVIĆ AND ROBINSON

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ABSTRACT. An algorithm is given for determining whether or not a finite
system of conditions of the types \( a < B, a > B, a \) is a square, possess a
simultaneous solution in positive integers. Various generalizations are also
obtained.

In 1970, Matijasevič proved that there is no algorithm to determine if a
general polynomial Diophantine equation has a solution in positive integers
(see, for example, Davis, Matijasević and Robinson [1]). A particularly neat
formulation of this theorem can be obtained from the observation made by
Skolem [6] that any Diophantine equation can be reduced to a system of
conditions of types \( a + \beta = \gamma, a \cdot \beta = \gamma \). The theorem then reads: There is
no algorithm to determine whether a system of conditions of types: \( a + \beta =
\gamma, a \cdot \beta = \gamma \) has a solution in positive integers.

There are other relations such that certain systems of conditions using
those relations are equivalent to \( a + \beta = \gamma \) and \( a \cdot \beta = \gamma \). For example,
consider the relations \( a + 1 = \beta, a \cdot \beta = \gamma \). Then, \( x + y = z \) is equivalent to
the system \( x + 1 = \alpha_1, z + 1 = \alpha_2, \alpha_1 \cdot \alpha_2 = \alpha_3, \alpha_3 + 1 = \alpha_4, \alpha_2 \cdot y = \alpha_5,
\alpha_5 + 1 = \alpha_6, \alpha_6 \cdot \alpha_4 = \alpha_7, \alpha_1 \cdot y = \alpha_8, \alpha_8 + 1 = \alpha_9, \alpha_9 \cdot \alpha_2 = \alpha_{10}, \alpha_{10} \cdot \alpha_2 =
\alpha_{11}, \alpha_{11} + 1 = \alpha_7 \). (This is simply an expansion of \( s(sx \cdot sz) \cdot s(y \cdot sz) = s(sz \cdot
sz \cdot s(sx \cdot y)) \) where \( s\alpha = \alpha + 1 \).) Consequently, there is no algorithm to
determine if a system of conditions of types: \( a + 1 = \beta, a \cdot \beta = \gamma \) has a
solution in positive integers.

Similar methods have been used to extend the theorem to various classes of
relations (see, for example, Robinson [4] and Schwartz [5]). In particular,
Kosovskii [3] showed that there is no algorithm to determine if a system of
conditions of types: \( a + \beta = \gamma, a | \beta, a = \square \) has a solution in positive
integers. (\( a = \square \) means that \( a \) is a perfect square.)

This result motivated the problem posed by Davis, Matijasević and Robin-
son [1]:

(\( \ast \)) Does there exist an algorithm to determine if a sequence of formulas of
types: \( a < \beta, a | \beta, a = \square \) has a solution in positive integers?

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This paper includes a general theorem concerning binary relations which has as a corollary an affirmative solution to (*).

In this paper, \( \alpha, \beta \) are either variables or positive integers and \( N \) is the set of positive integers. \( \alpha R \beta \) means "\( \alpha \) is related to \( \beta \) in relation \( R \)"

\[
R(\alpha) = \{ \beta | \alpha R \beta \}, \quad R(S) = \bigcup_{\alpha \in S} R(\alpha).
\]

Below, contexts will be encountered in which specific variables have their range restricted by stipulations already made. In such contexts, \( \langle \alpha \rangle \) represents the set of possible values of the variable \( \alpha \).

**Definition.** The family of computable functions \( \{ g(l): l \in N \} \) is called a *generalized common multiple in the relation \( R \)* if for any \( \alpha_1, \ldots, \alpha_i \),

\[
g^{(l)}(\alpha_1, \ldots, \alpha_i) \in \bigcap_{i=1}^l R(\alpha_i).
\]

Consider computable relations \( R_1, \ldots, R_m \) and computable sets \( S_1, \ldots, S_n \). Let

\[
P_f(\alpha) = R_f(\alpha) \cap S_f.
\]

Then we have:

**Main Theorem.** If (i) for all \( i = 1, \ldots, m \), \( \alpha R_i \beta \Rightarrow \alpha < f(\beta) \) for some strictly increasing computable function \( f \),

(ii) there is a family \( \{ g^{(l)}: l \in N \} \) which is a generalized common multiple in the relation \( P_f \).

For all \( i = 1, 2, \ldots, m \) there exists \( c_0 \) such that

(iii) either for all \( \alpha > c_0 \alpha R_i \alpha \), in which case \( R_i \) is called \( c_0 \)-reflexive, or for all \( \alpha > c_0 \sim \alpha R_i \alpha \), in which case \( R_i \) is called \( c_0 \)-antireflexive, and

(iv) for all \( \alpha > c_0 \), \( R_f(R_f(\alpha) \setminus \{ \alpha \}) \subseteq R_f(\alpha) \setminus \{ \alpha \} \),

then there exists an algorithm to determine whether or not a given system of conditions of the types \( \alpha R_i \beta, \alpha \in S_i \), has a solution in positive integers.

Let \( P \) be some given system consisting of \( p_0 \) conditions. The proof of the Main Theorem will use the following definitions, of which the second is inductive.

**Definition.** \( L_p \) is the set of (numbers or variables) \( \alpha \) such that \( P \) contains a sequence of conditions of the form:

\[
\alpha R_{i_1} \alpha_1, \quad \alpha_1 R_{i_2} \alpha_2, \quad \ldots, \quad \alpha_{w-1} R_{i_w} \alpha
\]

where some \( R_{i_j} \) is \( c_0 \)-antireflexive.

**Definition.**

\[
B_p = \{ c \in N: (\alpha R c) \in P \} \cup \{ \alpha: (\alpha R \beta) \in P \land \beta \in (L_p \cup B_p) \}.
\]

Thus, \( L_p \subseteq B_p \).

Let \( c_1 = \max_{c \in B_p} c \), \( c_2 = f^{(p_0)}(c_0) + f^{(p_0)}(c_1) \) where \( f \) and \( c_0 \) are from the
statement of the theorem, and $f^{(p_0)}$ is the function obtained by $p_0$ iterations of $f$:

**Definition.** \( P = \{ \alpha R \beta: (\alpha R \beta) \in P \land \alpha, \beta \in B_P \} \cup \{ \alpha \in S_i: (\alpha \in S_i) \in P \land \alpha \in B_P \} \).

The proof of the Main Theorem follows from the following.

**Lemma.** \( P \) has a solution in positive integers if and only if \( P \) has a solution in positive integers < \( c_2 \).

**Proof.** Suppose \( P \) has a solution. Let \( \alpha \in L_P \). Then there is the sequence of conditions (1) where \( R_j \) is \( c_0 \)-antireflexive. Now, suppose that \( P \) had a solution in which \( \alpha_{j-1} \) had a value \( x > c_0 \). Then \( x \notin \langle \alpha_j \rangle \). Therefore, \( \langle \alpha_j \rangle \subseteq R_u(x) - \{ x \} \), and

\[
x \in R^{(w-1)}(\langle \alpha_j \rangle) \subseteq R_u(x) - \{ x \},
\]

using (1) and (iv). This contradiction shows that for any solution, \( \alpha_{j-1} \leq c_0 \) and \( \alpha_i < f^{(p_0)}(c_0) \) for \( i = 1, 2, \ldots, w \). Hence any solution of \( P \) is such that all variables in \( L_P \) have values \( < f^{(p_0)}(c_0) \).

Next let \( \alpha \in B_P \). Then \( P \) contains a sequence of the form

\[
\alpha R_0 \alpha_1, \alpha_1 R_1 \alpha_2, \ldots, \alpha_w R_w \beta
\]

where \( w < p_0 \) and \( \beta \) is either a constant \( < c_1 \) or a variable in \( L_P \) and hence with value \( < f^{(p_0)}(c_0) \) in any solution. Then, in any solution of \( P \),

\[
\alpha < f^{(w)}(\max(c_1, f^{(p_0)}(c_0)))
= \max(f^{(w)}(c_1), f^{(w+p_0)}(c_0)) < c_2.
\]

Since all variables in \( B_P \) have values \( < c_2 \) in any solution of \( P \), \( P \) has such a solution.

Conversely, suppose \( P \) has such a solution. Then any "loop" of the form (1) in which no \( R_j \) is \( c_0 \)-antireflexive can be satisfied by

\[
\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_w > c_0.
\]

Consequently all such "loops" in \( P \) can be eliminated by replacing each occurrence of \( \alpha_i (i = 1, 2, \ldots, w) \) by \( \alpha \). (But the value to be assigned \( \alpha \) must be \( > c_0 \).

For the purpose of this proof \( \alpha \) is called a parent of \( \beta \) if the condition \( \alpha R_i \beta \) is in \( P \) for some \( i \), and the generation of \( \alpha \), for \( \alpha \notin B_P \), is the largest \( w \) such that \( P \) contains a sequence

\[
\alpha_1 R_{i_1} \alpha_2, \alpha_2 R_{i_2} \alpha_3, \ldots, \alpha_w R_{i_w} \alpha
\]

where \( \alpha_2 \notin B_P \). Since \( P \) has a solution we can assign values to all \( \alpha \in B_P \). Also assign all parentless \( \alpha \)'s the value \( s_0 \) where \( s_0 = \min(S_i) \). At this point any variable of the first generation, say \( \alpha \), has parents which are either constants or have already been assigned values. Suppose these parents have values \( a_1, \ldots, a_i \). Then fix the value of \( \alpha \) as \( g^{(i+1)}(a_1, \ldots, a_i, f(c_0)) \). (\( f(c_0) \) is included as an argument in order to guarantee, by (i) that variables arising
from collapsed "loops" are given values \( > c_0 \). In this way the whole first generation is assigned values. Now any variable of the second generation has parents with definite values. Continue this process until all variables have been assigned values. These values constitute a solution of \( P \) in positive integers.

Using the same notation as above and sacrificing some generality a much simpler statement of the Main Theorem can be obtained.

**Corollary.** If (i) there is a family \( \{ g^{(l)} : l \in N \} \) which is a generalized common multiple in the relation \( P_l \),

(ii) \( R_i \) is reflexive or antireflexive, and

(iii) \( \alpha R_i \beta \) implies \( \alpha < \beta \),

then there is an algorithm to determine whether or not a system of conditions of types \( \alpha R_i \beta, \alpha \in S_i \), has a solution in positive integers.

There are two interesting applications of this corollary.

The first gives a positive solution to the problem posed in [1].

**Corollary.** There is an algorithm to determine whether or not a system of conditions of types \( \alpha < \beta, \alpha \parallel \beta, \alpha = \square \) has a solution in positive integers.

The second finds a fine boundary line between decidable and undecidable problems.

**Theorem.** There is an algorithm to determine whether or not a system of conditions of type \( f_i(\beta) < \alpha \), where the \( f_i \) are any recursive, strictly increasing functions, has a solution in positive integers. However for general nondecreasing functions \( f_i \) there is no such algorithm.

**Proof.** The first assertion is an immediate consequence of the first corollary above where the \( R_i \) of the corollary are \( f_i(\beta) < \alpha \). To prove the second assertion, suppose there were such an algorithm. Then in particular there is an algorithm to determine whether or not there is a solution in positive integers to the system:

\[
f_1(\beta) - 1 < \alpha, \quad f_i^{-1}(\alpha) - 1 < \beta, \quad f_2(\gamma) - 1 < \alpha, \quad f_2^{-1}(\alpha) - 1 < \gamma
\]

where \( f_i^{-1}(\alpha) = \min_x(f_i(x) > \alpha) \). (If the minimum does not exist let \( f_i^{-1}(\alpha) \) be "infinite".) This sequence of formulas is equivalent to \( f_1(\beta) = \alpha = f_2(\gamma) \). Consequently, if there were an algorithm to determine whether or not this system has a solution then there would be an algorithm to determine whether or not \( \text{Range}(f_1) \cap \text{Range}(f_2) = \emptyset \). But since every computable set—and, in particular, every context-free set—is expressible as \( \text{Range}(f) \) for some increasing computable function \( f \), we would then have an algorithm to determine whether \( L(\Gamma_1) \cap L(\Gamma_2) = \emptyset \) where \( \Gamma_1 \) and \( \Gamma_2 \) are context-free grammars and where \( L(\Gamma) \) is the language accepted by \( \Gamma \). No such algorithm exists (cf., e.g., [2, p. 583]).
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References


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