PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 3, December 1979

A UNIQUENESS THEOREM FOR A BOUNDARY VALUE PROBLEM

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ABSTRACT. In this paper it is proved that the two-point boundary value problem, namely $(d^{(4)}/dx^4 + f)y = g$, $y(0) - A_1 = y(1) - A_2 = y''(0) - B_1 = y''(1) - B_2 = 0$, has a unique solution provided $\inf_x f(x) = -\eta > -\pi^4$. The given boundary value problem is discretized by a finite difference scheme. This numerical approximation is proved to be a second order convergent process by establishing an error bound using the L_2-norm of a vector.

1. Introduction. Consider the real two-point linear boundary problem

$$Ly \equiv \left[d^{(4)}/dx^4 + f(x) \right] y = g(x), \qquad 0 < x < 1,$$
$$y(0) = A_1, \quad y(1) = A_2, \quad y''(0) = B_1, \quad y''(1) = B_2, \tag{1}$$

where the functions $f(x)$ and $g(x) \in C[0, 1]$. A more general problem of the form

$$Ly = g(x), \quad y(a) = \bar{A}_1, \quad y(b) = \bar{A}_2, \quad y''(a) = \bar{B}_1, \quad y''(b) = \bar{B}_2$$

can always be transformed into (1) by means of a substitution of the form $X = (x - a)/(b - a)$. Problems of the form (1) frequently occur in plate deflection theory (see Reiss et al. [6]). The analytical solution of (1) is given by Timoshenko and Woinowsky-Krieger [7] provided the functions $f(x)$ and $g(x)$ are constants. In the general case we resort to some numerical techniques. Usmani and Marsden [8] have analyzed a second order convergent finite difference method for (1). Following this, Jain et al. [4] have developed and analysed higher order methods. The problem (1) does not always have a unique solution for all choices of $f(x)$ as is apparent from the example

$$y^{(4)} - \pi^4 y = 0, \qquad y(0) = y(1) = y''(0) = y''(1) = 0$$

which has as its solution $y(x) = C \sin(\pi x)$ for arbitrary values of C. The purpose of this note is to establish a sufficient condition that guarantees a unique solution for (1).

2. A uniqueness theorem. We shall give an elementary proof of the following theorem.

Received by the editors June 6, 1978.

AMS (MOS) subject classifications (1970). Primary 34B05; Secondary 65L10.

Key words and phrases. Brouwer's fixed point theorem, Euclidean and L_2-norm of a vector, finite difference scheme, local truncation error, plate deflection theory, posteriori error bound, spectral norm of a matrix, two-point boundary value problem.

THEOREM 1. *The boundary value problem* (1) *has a unique solution provided*

$$\inf_x f(x) = -\eta > -\pi^4, \quad \text{that is } -f(x) < \eta. \tag{2}$$

We preface the proof of this theorem with the following lemmas.

LEMMA 2. *If* $y(x) \in C^1[0, 1]$ *and* $y(0) = y(1) = 0$, *then*

$$\pi^2 \int_0^1 y^2(x)\, dx \leq \int_0^1 [y'(x)]^2\, dx.$$

Let $C[0, 1]$ consist of all continuous functions on the interval $I = [0, 1]$ and define in this section only $\|y\| = \sup_x |y(x)|$, $x \in I$.

LEMMA 3. *If* $y(0) = y(1) = 0$ *and* $y(x) \in C[0, 1]$, *then*

$$\|y\| \leq 0.5\left[\int_0^1 [y'(x)]^2\, dx\right]^{1/2}.$$

For the proofs of these lemmas the reader should consult Hardy et al. [**2**, Theorem 256, p. 182] and Lees [**5**].

LEMMA 4. *For the differential system*

$$Ly = g(x), \qquad y(0) = y(1) = y''(0) = y''(1) = 0,$$
$$\|y\| \leq 0.5\pi\|g\|/[\pi^4 - \eta].$$

PROOF. The system

$$\text{(i)} \qquad y''(x) = z(x), \qquad y(0) = y(1) = 0,$$
$$\text{(ii)} \qquad z''(x) + f(x)y = g(x), \qquad z(0) = z(1) = 0 \tag{3}$$

is equivalent to the differential system of the theorem. On multiplying (3.i) by $y(x)$ and integrating the result from 0 to 1, we find

$$-\int_0^1 (y')^2\, dx = \int_0^1 yz\, dx.$$

Now using the Cauchy-Schwartz inequality we obtain from the preceding equation

$$\int_0^1 (y')^2\, dx \leq \left[\int_0^1 y^2\, dx\right]^{1/2}\left[\int_0^1 z^2\, dx\right]^{1/2}.$$

On using Lemma 2, we derive from the preceding inequality

$$\left[\int_0^1 (y')^2\, dx\right]^{1/2} \leq \frac{1}{\pi^2}\left[\int_0^1 (z')^2\, dx\right]^{1/2}. \tag{4}$$

In a similar manner, from (3.ii), we derive

$$\left[\int_0^1 (z')^2\, dx\right]^{1/2} \leq \pi^3 \frac{\|g\|}{[\pi^4 - \eta]} \tag{5}$$

provided η satisfies (2). Now from (4) and (5) it follows that

$$\left[\int_0^1 (y')^2\,dx\right]^{1/2} \leq \pi \frac{\|g\|}{[\pi^4 - \eta]}. \tag{6}$$

Lemma 4 now follows from (6) and Lemma 3.

PROOF OF THEOREM 1. Assume that there exist two distinct functions $u(x)$ and $v(x)$ satisfying (1). Then it is easily seen that $\phi(x) = u(x) - v(x)$ satisfies

$$L\phi = 0, \qquad \phi(0) = \phi(1) = \phi''(0) = \phi''(1) = 0. \tag{7}$$

Now, from Lemma 4 and (7) it follows that $\|\phi\| \leq 0$, which proves $\|\phi\| \equiv 0$ and $u(x) \equiv v(x)$, $x \in I$. This proves that the boundary value problem (1) has at most one solution.

In order to prove that (1) indeed has a solution, we define functions $y_i(x)$, $i = 1, \ldots, 4$, as solutions of the respective initial value problems.

$$\begin{array}{llll}
\text{(i)} & Ly_1 = g(x), & y_1(0) = A_1, & y_1'(0) = y_1''(0) = y_1'''(0) = 0,\\
\text{(ii)} & Ly_2 = 0, & y_2'(0) = 1, & y_2(0) = y_2''(0) = y_2'''(0) = 0,\\
\text{(iii)} & Ly_3 = 0, & y_3''(0) = B_1, & y_3(0) = y_3'(0) = y_3'''(0) = 0,\\
\text{(iv)} & Ly_4 = 0, & y_4'''(0) = 1, & y_4(0) = y_4'(0) = y_4''(0) = 0.
\end{array} \tag{8}$$

From the continuity of $f(x)$ and $g(x)$ we are assured that unique solutions of these initial value problems exist on [0, 1]. Furthermore the function $z(x) \equiv z(x, s, t) = y_1 + sy_2 + y_3 + ty_4$, s, t being scalars, satisfies the initial value problem

$$Lz = g(x), \qquad z(0) = A_1, \qquad z'(0) = s, \qquad z''(0) = B_1, \qquad z'''(0) = t.$$

The function $z(x)$ will be a solution of (1) provided s, t satisfy

$$sy_2(1) + ty_4(1) = A_2 - y_1(1) - y_3(1),$$
$$sy_2''(1) + ty_4''(1) = B_2 - y_1''(1) - y_3''(1).$$

If $\Delta = y_2(1)y_4''(1) - y_2''(1)y_4(1) \neq 0$, a unique solution of the preceding linear system can be found, say s^*, t^*, and the corresponding function $z(s, s^*, t^*)$ then is the unique solution of (1). However, if $\Delta = 0$, then

$$y_2(1)/y_2''(1) = y_4(1)/y_4''(1) = p \text{ (constant)}.$$

We can assume that $p \neq 0$, because if $p = 0$, then $y_2(1) = 0$ and the solution of

$$Ly_2 = 0, \qquad y_2(0) = y_2''(0) = y_2'''(0) = y_2(1) = 0$$

from Taylor series has the property that $y_2'(0) = 0$, contradicting the original assumption that $y_2'(0) = 1$. Similarly p cannot be unbounded. Thus it follows that $y_2(1) = py_2''(1), p < \infty$.

Now using the system (8.ii), and the Taylor series, we obtain

$$y_2(1) = 1 - \tfrac{1}{24}f(\alpha)y_2(\alpha), \qquad 0 < \alpha < 1,$$
$$y_2''(1) = -0.5f(\beta)y_2(\beta), \qquad 0 < \beta < 1. \tag{9}$$

On combining $y_2(1) = py_2''(1)$ with equations (9) we obtain

$$f(\alpha)y_2(\alpha) - 12pf(\beta)y_2(\beta) = 24,$$

for all $f(x) \in C$. In an attempt to determine $y_2(\alpha)$ and $y_2(\beta)$, we choose $f(x) \equiv 1$ and $f(x) \equiv -1$, giving the system

$$y_2(\alpha) - 12py_2(\beta) = 24, \qquad -y_2(\alpha) + 12py_2(\beta) = 24.$$

But this latter system in the unknowns $y_2(\alpha)$ and $y_2(\beta)$ is inconsistent. We thus conclude that Δ cannot vanish and the proof of the Theorem 1 is completed.

3. A discrete boundary value problem. Let N be a positive integer and $h = (N+1)^{-1}$. We define the grid points $x_n = a + nh$, $n \in \{0, N+1\} \cup S$ where $S = (1, 2, \ldots, N)$. We denote by Φ the set of all real-valued functions defined on $\{x_n\}$, $n \in S$. Clearly Φ is a real linear space of dimension N. Also let $\|u\| = [\Sigma_i hu_i^2]^{1/2}$, where $u_i \equiv u(x_i)$. Note that $\|\cdot\|$ defines the L_2-norm of a vector, a natural definition of a norm on vectors since this norm converges to $[\int_0^1 u^2(x)\,dx]^{1/2}$ as $h \to 0$. We also have $\|u\| = \sqrt{h}\,\|u\|_2$ where $\|\cdot\|_2$ is the Euclidean norm (see Isaacson and Keller [3]). For a given matrix $A = (a_{ij})$, the matrix norm induced by the Euclidean vector norm we define the Hilbert or spectral norm of a matrix by $\|A\|_2 = \sqrt{\mu}$ where μ is the largest eigenvalue of A^*A. Here the operation * denotes the conjugate transpose of a matrix.

We now discretize the problem (1) by the following finite difference scheme

$$\text{(i)} \quad -2y(x_0) + 5y(x_1) - 4y(x_2) + y(x_3) = -h^2y''(x_0) + h^4\left[-\tfrac{1}{12}y^{(4)}(x_0) + y^{(4)}(x_1)\right] + t_1,$$

$$\text{(ii)} \quad \delta^4y(x_n) = h^4y^{(4)}(x_n) + \tfrac{1}{6}h^6y^{(6)}(\omega_n), \qquad n = 2, \ldots, N-1, \quad x_{n-2} < \omega_n < x_{n+2},$$

$$\text{(iii)} \quad y(x_{N-2}) - 4y(x_{N-1}) + 6y(x_N) - 2y(x_{N+1}) = -h^2y''(x_{N+1}) + h^4\left[y^{(4)}(x_N) - \tfrac{1}{12}h^4y^{(4)}(x_{N+1})\right] + t_N, \tag{10}$$

where $t_i = \frac{59}{360}h^6y^{(6)}(\omega_i)$, $i = 1, N$, $x_0 < \omega_1 < x_3$, $x_{N-2} < \omega_N < x_{N+1}$. Set $Y = (y_n)$ where y_n is an approximation to $y(x_n)$, $y(x)$ being the exact solution of (1). As in [4], [8], we obtain, on neglecting the local truncation errors t_n, noting $y^{(4)} = -f(x)y + g(x)$ and $y(x_n) \simeq y_n$,

$$P^2Y = -h^4DY + C, \qquad P^{-1} > 0, \tag{11}$$

(see [8]) where the tridiagonal matrix $P = (p_{ij})$ is given by $p_{ii} = 2, p_{ij} = -1$ for $|i - j| = 1$, otherwise $p_{ij} = 0$; $D = \operatorname{diag}(f_n)$ is a diagonal matrix and the column vector C depends on $g(x)$ and the boundary conditions. The matrix P is symmetric and positive definite and it is known that its eigenvalues are $4\sin^2(m\pi h/2)$, $m \in S$. Thus the eigenvalues of P^2 are

$$\lambda_m = 16\sin^4(m\pi h/2), \qquad m \in S. \tag{12}$$

LEMMA 5. $\pi^4 h^4(1 - \pi^2 h^2/6) \leqslant \lambda_1 \leqslant \pi^4 h^4$.

The inequality follows from $\theta - \theta^3/6 \leqslant \sin\theta \leqslant \theta$ for $0 < \theta < \pi/2$ and $(1 - x)^n > 1 - nx$ for small values of x. Also the eigenvalues satisfy

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N. \tag{13}$$

Since P is symmetric, it is easy to see that

$$\|P^{-2}\|_2 = 1/\lambda_1. \tag{14}$$

LEMMA 6. *Assume that $f(x)$ satisfies* (2) *and that h_0 is such that*

$$\eta < \pi^4(1 - \pi^2 h_0/6). \tag{15}$$

Furthermore if $h < h_0$, and $u, v \in \Phi$ satisfy

$$P^2 u = -h^4 Du + C_1, \qquad P^2 v = -h^4 Dv + C_2,$$

then $\|u - v\| \leqslant K(h_0)\|C_1 - C_2\|$, where

$$K(h_0) = h^{-4}\left[\pi^4(1 - \pi^2 h_0^2/6) - \eta\right]^{-1}. \tag{16}$$

PROOF. From the hypothesis it follows

$$\begin{aligned} P^2(u - v) &= -h^4 D(u - v) + (C_1 - C_2), \\ (u - v) &= P^{-2}\left[-h^4 D(u - v) + (C_1 - C_2)\right], \\ \|u - v\|_2 &\leqslant (1/\lambda_1)\left[\eta h^4 \|u - v\|_2 + \|C_1 - C_2\|_2\right], \end{aligned}$$

by (2) and (14) or

$$(\lambda_1 - \eta h^4)\|u - v\| \leqslant \|C_1 - C_2\|.$$

Now on using Lemma 5 and (15), the result of Lemma 6 follows.

REMARK. If $\eta = 0$, the constant $h_0 < 0.77$.

LEMMA 7. *If $f(x)$ satisfies* (2) *and if Y is a solution of* (11), *then*

$$\|Y\| \leqslant K(h_0) \cdot \|C\|. \tag{17}$$

PROOF. Put $u = Y$, $C_1 = C$, $v = C_2 = 0$ in Lemma 6, then (17) follows.

THEOREM 2. *If $f(x)$ satisfies* (2), *then the discrete boundary value problem* (11) *has a unique solution.*

PROOF. Clearly, Lemma 6 implies that (11) has at most one solution. Let $\Omega = \{u \in \Phi: \|u\| \leqslant K(h_0)\|C\|\}$. Define a mapping $T: u \to v$ by means of the relation

$$P^2 v = -h^4 Du + C. \tag{18}$$

Since $P^{-1} > 0$, it follows that (18) has exactly one solution for a given u.

Consider $Tu = v$ and use (18) to deduce

$$\begin{aligned}\|v\| &\leq \left[h^4\eta\|u\| + \|C\|\right]/\lambda_1\\ &\leq \left[\left(h^4\eta K(h_0) + 1\right)\|C\|\right]/\left[\pi^4h^4(1 - \pi^2h_0^2/6)\right]\\ &\leq K(h_0)\|C\|,\end{aligned}$$

on using (16). This proves that T maps Ω into itself. Let $\varepsilon > 0$ be given, we can choose $\delta(\varepsilon)$

$$\delta = \left[\varepsilon\pi^4\left(1 - \pi^2h_0^2/6\right)\right]/\eta, \qquad \eta \neq 0. \tag{19}$$

Now if $Tu_1 = y_1$, $Tu_2 = y_2$, then

$$\begin{aligned}\|Tu_1 - Tu_2\| &= \|y_1 - y_2\|\\ &= \left\|P^{-2}\left(-h^4Du_1 + C\right) - P^{-2}\left(-h^4Du_2 + C\right)\right\|\\ &\leq h^4\eta\|u_1 - u_2\|/\lambda_1 < \varepsilon,\end{aligned}$$

provided $\|u_1 - u_2\| < \delta$ given by (19) and $\lambda_1 > \pi^4h^4(1 - \pi^2h_0^2/6)$. This shows that T is continuous on Ω. Hence, by Brouwer's fixed point theorem [**1**], there is a $u \in \Omega$ such that $Tu = u$, and this is clearly a solution of (18) and hence of (11). This completes the proof of the theorem.

Note. For $\eta = 0$, an obvious modification of the argument still proves the preceding theorem.

4. An approximation theorem. In this concluding section we establish an a posteriori bound. We note that the system of linear equations based on (10) can be written as

$$P^2\tilde{Y} = -h^4D\tilde{Y} + C + T \tag{20}$$

where $\tilde{Y} = (y(x_n)) \in \Phi$ and clearly

$$\|T\| \leq \tfrac{1}{6}h^6M_6 \tag{21}$$

where $M_6 = \max_x|d^{(6)}y/dx^6|$, $0 \leq x \leq 1$. If we subtract (11) from (20), we obtain an error equation, namely

$$P^2E = -h^4DE + T \tag{22}$$

where $E = (e_n) \in \Phi$ and $e_n = y(x_n) - y_n$.

THEOREM 3. *If $f(x)$ satisfies* (2), *then for $h \leq h_0$*:

$$\|E\| = O(h^2).$$

PROOF. From Lemma 7, it follows that

$$\|E\| \leq K(h_0)\|T\| = O(h^2)$$

using (16) and (22). In fact

$$\|E\| \leq \tfrac{1}{6}M_6h^2\left[\pi^4\left(1 - \pi^2h_0^2/6\right) - \eta\right]^{-1}.$$

ACKNOWLEDGEMENT. The author acknowledges the National Research Council of Canada's financial support.

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